Attribution and Compensation Design in Online Advertising*

Yunhao Huang

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Abstract

This paper studies how the attribution algorithms used in online ad auctions affect the strategic interactions between advertisers and publishers, and it investigates optimal attribution strategies for advertisers. Because online advertisers typically advertise with several publishers to increase their reach, users may be exposed to ads from multiple publishers before converting. The attribution challenge for an advertiser is to measure the contributions of each publisher's advertising on conversions. These attributed conversion measures are crucial because they serve as inputs into the algorithms that advertisers use to determine bids in future ad auctions. The attribution challenge is aggravated by the fact that publishers typically have access to more information than advertisers, such as user behavior on their sites. This information asymmetry can lead to a moral hazard problem: publishers can exploit their information advantage to target ads to users who are likely to result in attributed conversions, rather than to users with large incremental ad effects. To investigate this misalignment of interests between advertisers and publishers, I cast the attribution problem as an incentive design problem. Using a structural model, I first characterize the dynamic incentives created by standard attribution algorithms and derive the advertiser's optimal strategy. I find that the advertiser's optimal strategy takes the form of team incentives, where each publisher is compensated only when a conversion is preceded by an ad impression by only that publisher. Counterfactual analysis shows that the optimal strategy increases the advertiser's ROI on the order of 20–40% compared with standard attribution algorithms. The findings highlight the importance of considering the dynamic incentives that measurement tools generate.

Keywords: Automated Bidding, Field Experiments, Moral Hazard, Online Advertising.

*Haas School of Business, University of California, Berkeley. Email: huangyh@haas.berkeley.edu. I am indebted to my advisors, Steven Tadelis, J. Miguel Villas-Boas, Zsolt Katona, and Sarah Moshary for their continuous guidance and support. I thank Stephen Anderson, Ron Berman, Xin Chen, Mark Dean, Kwabena Donkor, Soheil Ghili, Samuel Goldberg, Liang Guo, Elisabeth Honka, Sylvia Hristakeva, Yufeng Huang, Ganesh Iyer, Przemysław Jeziorski, Zhenling Jiang, Justin Johnson, Yuichiro Kamada, Kei Kawai, T. Tony Ke, Tongil "TI" Kim, Yewon Kim, Vineet Kumar, Bradley Larsen, Chenxi Liao, Jūra Liaukonytė, Song Lin, Michelle Y. Lu, Samir Mamadehussene, Moritz Meyer-ter-Vehn, Ilya Morozov, Sridhar Narayanan, Olivia Natan, A. Yeşim Orhun, Matthew Osborne, Omid Rafieian, Federico Rossi, Robert Sanders, Katja Seim, Mengze Shi, Zijun (June) Shi, Andrey Simonov, Andrzej (Andy) Skrzypacz, Sofia B. Villas-Boas, Yuyan Wang, Nils Wernerfelt, Alexander White, Yunfei (Jesse) Yao, Florian Zettelmeyer, Fan Zhang, and audiences at 2023 European Winter Meeting of the Econometric Society, International Industrial Organization Conference 2024, Marketplace Innovation Workshop 2024, Stanford Institute for Theoretical Economics 2024, and several seminars for helpful comments. I am grateful to the Management of Technology Fund and the Fisher Center for Business Analytics for funding research expenses. I was not compensated by the collaborating companies or by any other entity for the production of this article. I thank the collaborating companies for their efforts during the implementation of this study. All errors are my own.

1 Introduction

Firms are increasingly adopting algorithms to automate decision-making across various domains, such as bidding in online advertising auctions and pricing in online marketplaces. While the use of such automated decision-making algorithms has become widespread, less attention has been paid to their implications on strategic interactions between decision-makers and other market participants. This paper focuses on the strategic interactions that arise when a decision-maker utilizes algorithms to assign credit to a team of agents in dynamic settings, particularly in the context of online display advertising.

Online display advertising exemplifies this challenge of algorithmic decision-making in strategic environments. Advertisers face millions of advertising opportunities per second and rely heavily on automated bidding algorithms to participate in advertising auctions.[1] These algorithms are based on the intuition that more effective advertising should warrant higher bids. However, measuring the effectiveness of advertising is a complex task, so advertisers typically rely on heuristic methods. These methods are known as *attribution algorithms* because advertisers typically advertise across multiple publishers, and determining the effectiveness of advertising in this case typically involves credit-giving.[2] As an example, one of the most prevalent attribution strategies is the *last-touch* attribution. Under this algorithm, when a user is exposed to an advertisement on multiple publishers and subsequently converts (e.g., visits the advertiser's website or makes a purchase), the last publisher in the display sequence receives all the credit for the conversion. The attribution algorithm then has consequences for future advertising: publishers who received higher credit in the past are regarded as more effective and are more likely to receive higher bids from the advertiser in the future.

A crucial nuance in the application of attribution algorithms is potential information asymmetries between advertisers and publishers. Privacy regulations,[3] practical limitations,[4] and strategic information withholding[5] may result in publishers possessing user information that advertisers cannot obtain. For instance, granular data on user activities within a publisher's platform is rarely shared with advertisers. As a result, publishers may exploit their informational advantage to strategically target advertisements to users who are likely to result in attributed conversions, rather than to users where the adver-

[1] Major online advertising platforms, such as Facebook Ads and Google Ads, offer such automated bidding strategies to advertisers, which are typically set as the default bidding strategies.

[2] These algorithms encompass both credit assignment and bidding, and this paper refers to the entire procedure as attribution algorithms.

[3] Payment in exchange for data from the publisher to the advertiser can be classified as "selling data," which is subject to stringent privacy regulations (e.g., European Parliament and Council of the European Union, 2018; California Privacy Rights Act, 2020). See, for example, Ke and Sudhir (2022) and Johnson et al. (2023) for more discussion.

[4] In online display advertising auctions, user information is often transmitted by an HTTP bid request (IAB Technology Lab, 2016). To promote efficient transactions, industry standards encourage keeping bid request sizes relatively small, ideally within a few to tens of kilobytes.

[5] See, for example, Marotta et al. (2022) and D'Annunzio and Russo (2023) for more discussion.

tisement would have the most incremental effect. By doing so, a publisher could inflate its perceived advertising effectiveness, leading advertisers to submit higher bids to them in the future, even if the incremental value of the publisher's display is low.

To illustrate how this moral hazard problem can arise from attribution algorithms, consider the following toy example, where an advertiser adopts last-touch attribution to determine future bids. A publisher has two user segments, represented by users A and B, and the sizes of the two segments are the same. Without the publisher's display, user A's conversion rate is 100%, and user B's is 0%. If the publisher advertises to these users, user A's conversion rate remains 100%, and user B's increases to 50%. That is, the advertisement only has incremental effects for user B but not for user A. Moreover, this publisher will be the last in the display sequence if it displays the advertisement.

This toy example spans two periods. The user segment is observed by the publisher but not by the advertiser, so in each period, the advertiser can only submit a single bid, but the publishers can decide which segment(s) to display. The bid submitted by the advertiser depends on its perceived advertising effectiveness, attributed conversion rate r. One of the most common bidding rules is *target cost-per-action (CPA)* bidding, that is, the bid is the anticipated cost per conversion action multiplied by the attributed conversion rate.[6] Suppose that the advertiser adopts this bidding rule with a target CPA of \$4, then the advertiser's bid b in period t is $b_t = \$4 \times r_t$. In the first period, the advertiser has no data, leading to an uninformative prior that r_1 is in $[0, 1]$. In the second period, the advertiser may observe the conversion data in the first period, and r_2 can be calculated according to the data, assuming a weak prior.

The advertiser's bid in the second period depends on the publisher's advertisement targeting decision in the first period. If the publisher displays to user A only, the attributed conversion rate would be 100%, leading to a bid of \$4; if the publisher displays to both users A and B, the attributed conversion rate would be 75%, leading to a bid of \$3; if the publisher displays to user B only, the attributed conversion rate would be 50%, leading to a bid of \$2; if the publisher displays to no one, the attributed conversion rate would be the same as the uninformative prior, leading to a bid of \$2. Note that from the advertiser's perspective, it could be optimal to display to user B, as the advertisement only has an incremental effect on user B. However, from the publisher's perspective, its optimal choice could be to display to user A.[7]

This example underscores the close connection between attribution algorithms and publisher incentives. The dynamic incentives generated by attribution algorithms, combined with information asymmetry between advertisers and publishers, can lead to strategic manipulation by publishers. The publisher's actions, driven by the goal of receiving credit for conversions are misaligned with the advertiser's goal of generating incremental

[6] See Appendix A.1 for a micro-foundation of this bidding rule.
[7] See Appendix A.2 for details on the publisher's control over the display of advertisements.

conversions. As a result, the advertiser may be misled into believing that the campaign is highly effective, leading to increased bids based on this mistaken advertising effectiveness measurement.

In practice, there is a large class of attribution algorithms, and they all utilize the *touchpoint sequence*, that is, the sequence of publishers displaying the advertisement.[8] These algorithms differ in how they assign credit based on the touchpoint sequence. The use of various attribution algorithms leads to the first research question addressed in this study: what incentives do attribution algorithms create for publishers? As illustrated by the motivating example, these algorithms may incentivize publishers to display advertisements to users whose conversions are more likely to be attributed to them. Furthermore, different attribution algorithms could create distinct incentives, potentially leading to varied publisher behaviors and outcomes for advertisers.

To examine the incentives created by attribution algorithms, I develop a model that captures the dynamic interactions between advertisers and publishers. The model incorporates a primitive that is novel in online advertising settings, publishers' opportunity costs to display the advertisement. This opportunity cost captures a publisher's cost-benefit analysis regarding whether to display the advertisement, accounting for the value of displaying other advertisements and other factors such as reputation costs. Moreover, users are heterogeneous and publishers have more information about user characteristics than advertisers.

The dynamic model reveals that the incentives provided to publishers by attribution algorithms can be replicated by a static incentive scheme, which I refer to as the *static equivalent*. This implies that the dynamic interactions between advertisers and publishers can be replicated by a static model with appropriately constructed incentives. These static incentives resemble contracts and can be implemented using *pay-per-action* mechanisms with contingent actions, where the contingent actions are based on the touchpoint sequence and the conversion outcome, and publishers are compensated according to these contingent actions.[9] By characterizing the static equivalents of various attribution algorithms, I provide a framework for comparing the incentives they create.

Having characterized the static equivalents of attribution algorithms, I subsequently investigate the second research question, the optimal attribution strategy for the advertiser. The static equivalence results enable me to approach this question by focusing on the static contract-like incentives and optimizing advertiser profits over all such incentives. It is worth mentioning that these static incentives can be directly implemented through pay-per-action mechanisms, so studying these static incentives *per se* could also be valuable.

[8] While touchpoints can also encompass other forms of user behavior such as clicks, this study specifically focuses on impressions, hence the term touchpoint is used to denote impressions in this study.

[9] See Appendix A.3 for a detailed discussion of the implementation.

To address the question of optimal incentives, I assume and subsequently validate empirically that the opportunity costs across publishers are *affiliated*, a form of positive correlation as defined by Milgrom and Weber (1982). Affiliation implies that a low opportunity cost for one publisher is likely indicative of low opportunity costs for other publishers. Under affiliated opportunity costs, the optimal strategy for the advertiser is to define, for each publisher, a Conversion Action as a conversion preceded by an advertisement impression by only that publisher. The advertiser pays that publisher only when this Conversion Action happens. To illustrate, consider a scenario with two publishers, 1 and 2. The advertiser can define Conversion Action I as an event where a user views the advertisement exclusively on publisher 1 and subsequently visits the advertiser's website. The advertiser then compensates publisher 1 when this conversion action occurs. Similarly, the advertiser can define Conversion Action II as an event where a user views the advertisement exclusively on publisher 2 and subsequently visits the advertiser's website. The advertiser then compensates publisher 2 accordingly. The intuition is that when multiple publishers display the advertisement to a user, the opportunity costs are likely to be low, and thus, the advertiser need not pay a high amount to either publisher for the display.

The theoretical insights call for an empirical investigation. First, the results demonstrate that the characteristics of model primitives play a crucial role in determining the form of optimal incentives. Therefore, it is important to examine these characteristics using real-world data. Second, an empirical analysis can provide a quantitative assessment of the effectiveness of the optimal incentives and attribution algorithms, offering valuable insights for practical implementation.

To conduct the empirical analysis, I collaborate with an unnamed advertiser to conduct an online advertising experiment. In the experiment, I submit bids to two major publishers on a major advertising exchange, with the bids randomized at the auction level. This randomization allows for the identification of the model parameters. After collecting the experimental data, I estimate the model using maximum likelihood estimation. I then simulate counterfactual profits under various strategies, including the optimal strategy and other standard attribution algorithms.

The results suggest that advertisers can achieve substantial gains by adopting the optimal incentive strategy. Specifically, the optimal strategy increases the advertiser's return on investment (ROI) on the order of 20–40% compared with standard attribution algorithms. Moreover, I find that *single-touch* attribution algorithms, which include first-touch and last-touch attribution, yield higher profits than *multi-touch* attribution algorithms, such as linear, causal, and Shapley attribution. The intuition of this finding is that the incentives created by single-touch attribution algorithms and the optimal strategy are more closely aligned. When a touchpoint sequence involves multiple publishers, the optimal strategy assigns credit to none of them, while single-touch attribution algorithms assign credit to exactly one publisher. In contrast, multi-touch attribution algorithms assign credit to all publishers involved. Consequently, models that assign credit based on marginal causal effects, such as the causal and Shapley attribution algorithms, deviate further from the optimal incentives for publishers. These findings underscore the importance of considering the incentives generated by measurement tools when designing incentive schemes.

This research is related to three strands of literature. The first strand of literature pertains to the measurement of advertising effectiveness (e.g., Lewis et al., 2011; Blake et al., 2015; Gordon et al., 2019). This body of work highlights the difficulties in using observational data to measure advertising effectiveness. It points out the potential endogeneity issues resulting from correlated user behavior across websites, the "activity bias." This bias in estimating advertising effectiveness has led to significant efforts devoted to developing superior measurement technologies (e.g., Li and Kannan, 2014; Xu et al., 2014; Barajas et al., 2016; Du et al., 2019). This paper takes a step further by exploring the link between incentives and measurement. I show a novel source of bias stemming from publishers' strategic advertisement display decisions. I characterize how an advertiser's use of attribution algorithms may create dynamic incentives and how observational data can be leveraged to manage such incentives. The analysis of publishers' responses to different attribution algorithms distinguishes this study from previous work.

The second strand of literature examines the impact of attribution methods on advertiser profits (Li et al., 2016; Abhishek et al., 2017; Berman, 2018; Danaher and van Heerde, 2018). This literature emphasizes that naïve attribution methods can lead to inefficient decisions, such as suboptimal bidding strategies. This paper contributes by studying two novel forms of touchpoint utilization, attribution algorithms and pay-per-action strategies with conversion actions that are contingent on touchpoint sequences. It innovates by considering the incentives offered to publishers, particularly the dynamic incentives associated with attribution algorithms. While dynamic incentives have been studied extensively in the contexts of pricing (e.g., Fudenberg and Villas-Boas, 2006; Zhang, 2011) and salesforce compensation (e.g., Kuksov and Villas-Boas, 2019), this study contributes to the relatively underexplored field of dynamic incentives in online advertising and auction contexts.

The third strand of literature involves agency theory, wherein the optimal incentive scheme resembles the tournaments described in team compensation literature (Lazear and Rosen, 1981; Green and Stokey, 1983; Nalebuff and Stiglitz, 1983). This study adapts and extends insights from this literature to the online advertising environment. It casts online advertising as a team compensation problem, where advertisers are the principals and publishers are the agents. It then investigates how touchpoint data can be used as auxiliary information in the incentive design framework to partially address the information asymmetry issue between advertisers and publishers.

The structure of the rest of the paper is as follows. Section 2 develops a dynamic model and characterizes the incentives created by attribution algorithms. Section 3 studies the optimal strategy for the advertiser. Section 4 provides details about the dataset, explains the estimation procedure, and presents the outcomes of the estimation. Section 5 proceeds with the counterfactual analysis and showcases its results. Section 6 offers concluding remarks.

2 Attribution Algorithms and Dynamic Incentives

This section develops a dynamic model to characterize the incentives created by attribution algorithms. I first describe the stage game with a focal advertiser and multiple publishers and build the dynamic model based on it. Then I show how the dynamic incentives can be replicated by static contract-like incentives. In Appendix B.1, I consider an extension where the main strategic effects of the publishers are present in an equilibrium framework with multiple advertisers; for expositional purposes, I present the simpler model here.

2.1 Stage Game

The game consists of a focal advertiser and J publishers. Users are not explicitly included as players in this game, because this paper's focus is incentives of the publishers. Specifically, the publishers that a user browses and how touchpoints influence user conversion are treated as exogenous. Moreover, to simplify notation, henceforth everything is conditional on publicly observed user characteristics. In other words, the advertiser and the publishers are able to act differently for different user segments, and the model here is conditional on a specific user segment.

The timing of the stage game is as follows:

1. The advertiser submits a per-impression bid b_j to each publisher j.
2. A continuum of users arrives. The mass of users is normalized to one. Each user i has a browsing path H_i. As an example, if user i browses publishers $1, 2$ and 3, then $H_i = (1, 2, 3)$. Each publisher appears at most once in H_i.
3. The publishers on a user's browsing path decide simultaneously whether to display the advertisement to each user. If publisher j decides to display to user i, it incurs an *opportunity cost* of c_{ij}, detailed shortly, and receives b_j at the end of the period.
4. The touchpoint sequence T_i and conversion $Y_i \in \{0, 1\}$ of each user i are realized. As an example, if $H_i = (1, 2, 3)$, and publishers 1 and 3 display the advertisement

but publisher 2 does not, then $T_i = (1, 3)$. Note that both the browsing path H_i and the touchpoint sequence T_i are ordered sequences.

The opportunity costs for the publishers is an umbrella term. It could include (i) the value of outside options, such as displaying another advertisement, and (ii) the potential negative impact on the publisher's reputation because of a low-quality advertisement. When other advertisers' behavior and the incentives they create are held constant, part (i) can be seen as primitive.[10] The reputation cost in part (ii) is also fixed for a given advertisement.

Once the advertiser submits the bids $b = (b_j)$, the publishers participate in a Bayesian game, denoted as $\Gamma(b)$. Note that publishers simultaneously decide whether to display the advertisement or not. This simultaneous decision reflects the real-world scenario where publishers do not share information regarding whether they display advertisements to a given user. Consequently, a publisher does not observe other publishers' actions when making the display decision. Instead, a publisher would make Bayesian inferences about the behavior of other publishers, conditional on its available information, the browsing path H_i, and the opportunity cost c_{ij}. Note that c_{ij} is only fully observed by publisher j, but publisher j can infer the conditional distribution of $c_{ij'}$ for $j' \neq j$ in a Bayesian fashion. The opportunity cost vector for user i is denoted as $c_i \equiv (c_{ij})$, whose joint density distribution given the browsing path H_i is denoted as $f(c_i|H_i)$. Publisher j's advertisement display rule is denoted as a function a_j, mapping a user i's browsing path H_i and opportunity cost c_{ij} to whether the advertisement is displayed to user i.

It is worth discussing some simplifying assumptions made in the model.

First, each publisher appears at most once in the browsing path H_i. The reason for this assumption is that the main focus of this study is the interaction between the focal advertiser and multiple publishers, instead of frequent display decisions within a publisher. This applies to the case when the advertiser would like to reach a wider audience, that is, more unique users being exposed to the advertisement, so they have a low "frequency cap" on the number of displays to each user on each publisher. With this assumption, the set of user browsing sequences is vastly reduced and the intuition is sharpened. In my empirical application, it is ensured by bidding only once for each user on each publisher. Moreover, the main results in this section are still expected to hold when each publisher appears more than once in the browsing path, as similar dynamic incentives are expected to arise in that model.

Second, the publishers are well-informed about the browsing path H_i before making the display decisions. This assumption can be justified by the fact that the publishers

[10] See Appendix B.1 for an extension with multiple advertisers, which micro-founds part (i). In that model, the values and strategies of the advertisers are the primitives. In the model presented here, the values and behaviors of other advertisers affect the outcome only through the opportunity cost of the publisher. Hence, the opportunity cost of the publisher can be seen as primitive in the model.

possess a vast amount of data and can reasonably infer whether a user is multi-homing and the order of browsing paths. Specifically, publishers may utilize cross-site tracking tools, engage in data-sharing agreements, or participate in data marketplaces that allow them to access user data beyond their sites. When user identity is not perfectly tracked, publishers can employ sophisticated machine learning models to probabilistically link user activity across sites based on behavior patterns, device characteristics, and so on. There could be some random errors in publishers' inferences; to keep the model tractable, I push the assumption that publishers observe some signals to infer the browsing path to an extreme. It could be interesting for future research to identify from data how much information publishers have and investigate its implications on publisher incentives.

Moreover, I also assume that the advertiser observes the browsing path H_i and can submit different bids b_j for different browsing paths H_i; this assumption means that user browsing paths are a part of publicly observed user characteristics, and I henceforth conditional everything on the browsing path to simplify notation. This assumption can be justified by the fact that advertisers obtain user browsing path data in online advertising bid requests and can utilize user data to predict browsing paths.

2.2 Dynamic Model

Existing attribution algorithms operate on the heuristic that a higher attributed conversion rate on a publisher indicates higher advertising effectiveness, leading advertisers to increase future bids for that publisher. This heuristic may lead to dynamics in advertiser bidding and publisher display decisions.

Formally, the dynamic game G is an infinite repetition of the stage game among the publishers. In each period, each publisher determines to which users to display the advertisement, and their objective functions are their respective discounted payoffs, with δ as the common discount factor. The users in each period are short-lived, and their primitives are independent and identically distributed across periods.

Denote the advertiser's bid vector in period t as $b^{(t)}$, with $b_j^{(t)}$ being the bid for publisher j. In this setting, $b^{(t)}$ are payoff-relevant, so they are the state variables in period t. They are determined according to the following rules:

1. At the beginning of period 1, the advertiser sets the bid vector $b^{(1)}$ and sets an attribution algorithm. Specific classes of algorithms are discussed in Section 2.3.

2. For each period $t \in \mathbb{N}$, the stage game $\Gamma(b^{(t)})$ among the publishers is played: each publisher makes the advertisement display decisions, then the touchpoint sequence and conversions are realized, and the publishers receive payments according to $b^{(t)}$.

3. Define $\pi_{\omega,y}^{(t)} \equiv \mathbb{P}^{(t)}(T = \omega, Y = y)$ as the mass of users with touchpoint sequence ω and conversion y, and the vector $\pi^{(t)} \equiv (\pi_{\omega,y}^{(t)})$. At the end of period t, the

advertiser sets the bid vector for the next period, $b^{(t+1)}$, based on the bidding rule $\Psi : \mathbb{R}_+^J \times \mathbb{R}_+^{|\mathcal{T}|\times\{0,1\}} \to \mathbb{R}_+^J$, which takes $b^{(t)}$ and $\pi^{(t)}$ as inputs, that is,

$$b^{(t+1)} = \Psi(b^{(t)}, \pi^{(t)}). \tag{1}$$

Let us now discuss additional simplifying assumptions of the dynamic model.

First, the advertiser is assumed to be able to commit to the algorithm Ψ. In reality, major advertising platforms offer such algorithms as a service to the advertiser, even making it a default option. Once an advertiser selects a particular algorithm and sets the parameters, it typically refrains from intervening or manually adjusting the bids, relying instead on the algorithm's automated operations.

Second, in (1), I assume that the next period's bids are adjusted solely based on the bids and the data from the current period. A more general setting is that the advertiser could adjust bids based on data from more than one previous period, or make adjustments based on bids from more than one previous period, so that the inputs of the bidding rule Ψ may include $\pi^{(t-k)}$ and $b^{(t-k)}$ for $k = 1, \cdots, K$. However, this assumption simplifies the model, resulting in cleaner definitions and equilibrium characterizations. Moreover, in the stationary environment discussed later, the results are the same whether the advertiser uses data and bids from only the last period or the last $K > 1$ periods.

Third, I assume that each publisher j observes how the bids on it are adjusted dynamically, that is, it observes the bidding rule Ψ_j. This assumption can be justified given the frequency of advertisement auctions. It is feasible for a publisher to experiment with various advertisement display strategies and learn the advertiser's bidding rule based on the bids.

2.3 Attribution Algorithms

To make the model and analysis more concrete, in this subsection, I provide formal definitions of standard attribution algorithms by detailing the bidding rule Ψ.

An attribution algorithm includes two stages. First, having the distribution of touchpoints and conversions as input, the algorithm calculates an attributed conversion rate for each publisher. Second, it adjusts the bid for each publisher according to its attributed conversion rate. The key difference across attribution algorithms lies in the first step, the computation of the attributed conversion rates, which I discuss further in this subsection. I drop the time superscript to simplify the notation.

The commonly used methods to calculate attributed conversion rates can be grouped into two categories: *rule-based attribution methods* and *data-driven attribution methods*.[11] Rule-based attribution methods apply predefined rules to assign credit for conversions to

[11] The names of the two categories are widely used in the industry in such a way.

different publishers. In contrast, data-driven attribution methods use statistical models to quantify each publisher's impact on conversion rates and determine the corresponding credit.

Rule-based Attribution Methods Under these methods, when a user's sequence of touchpoints ω leads to a conversion, each publisher $j \in \omega$ is given a credit calculated by a weight function $\chi(j, \omega)$. Specific rules and their corresponding weight functions are:

- *Last-touch attribution*: the weight function $\chi^{\mathrm{LT}}(j, \omega) \equiv \mathbb{1}(j = \mathrm{last}(\omega))$.[12] In this case, if publisher j is the final touchpoint, it receives all the credit but receives none if it is not.
- *First-touch attribution*: the weight function $\chi^{\mathrm{FT}}(j, \omega) \equiv \mathbb{1}(j = \mathrm{first}(\omega))$. In this case, if publisher j is the initial touchpoint, it receives all the credit but receives none if it is not.
- *Linear attribution*: the weight function $\chi^{\mathrm{L}}(j, \omega) \equiv \dfrac{1}{|\omega|}$. All publishers in the touchpoints receive equal credit in this case.

The attributed conversion rate for publisher j is then given by[13]

$$r_j \equiv \mathbb{E}_{T,Y}\left[\chi(j, T) \cdot \mathbb{1}(Y = 1) | j \in T\right] = \frac{\sum_{(\omega,y):j\in\omega} \chi(j, \omega) \cdot \pi_{\omega,1}}{\sum_{(\omega,y):j\in\omega} \pi_{\omega,y}}. \tag{2}$$

The conditional expectation represents the probability of a conversion occurring ($Y = 1$), weighted by the function $\chi(j, T)$, conditional on that publisher j is in the sequence of touchpoints. The second equality is derived from Bayes' rule.

Data-driven Attribution Methods Under these methods, statistical models are employed to estimate the conversion rates conditional on touchpoints and user observables. Recall that everything in this section is conditional on user observables, so I omit them to ease notation. The first step is to estimate the conversion rate $\hat{\rho}_\omega$ conditional on each sequence of touchpoints ω, that is,

$$\hat{\rho}_\omega \equiv \frac{\pi_{\omega,1}}{\pi_\omega},$$

where

$$\pi_\omega \equiv \pi_{\omega,1} + \pi_{\omega,0}$$

[12] For instance, for $\omega = (1, 2)$, $\chi^{\mathrm{LT}}(1, \omega) = 0$ and $\chi^{\mathrm{LT}}(2, \omega) = 1$.

[13] For instance, consider users with browsing path $(1, 2)$. For publisher 1, there are four (ω, y) pairs such that $1 \in \omega$, that is, the combination of $\omega = (1, 2)$ or (1) and $y = 0$ or 1. Under last-touch attribution, publisher 1 receives the credit only when $\omega = (1)$ and $y = 1$, so $r_1 = \frac{\pi_{(1),1}}{\pi_{(1,2),1}+\pi_{(1,2),0}+\pi_{(1),1}+\pi_{(1),0}}$. In contrast, publisher 2 receives the credit when $\omega = (1, 2)$ or $\omega = (2)$ and $y = 1$, so $r_2 = \frac{\pi_{(1,2),1}+\pi_{(2),1}}{\pi_{(1,2),1}+\pi_{(1,2),0}+\pi_{(2),1}+\pi_{(2),0}}$.

is the mass of users with touchpoint ω, regardless of whether they convert or not. The next step is to calculate the incremental conversion rate resulting from publisher j's display of the advertisement, defined as

$$\Delta(j,\omega) \equiv \hat{\rho}_{\omega} - \hat{\rho}_{\omega\setminus\{j\}}.$$

In the causal inference framework, the quantity $\Delta(j,\omega)$ is known as an estimate of the *average treatment effect on the treated* and has an interpretation of the incremental effect of advertisement.[14] The (raw) *causal attribution* method computes the attributed conversion rate as a weighted average of the incremental conversion rates, with the weight being the probability of the sequence of touchpoints given that the publisher is in it:

$$r_j \equiv \mathbb{E}_T\left[\Delta(j,T)|j\in T\right] = \frac{\sum_{\omega:j\in\omega}\Delta(j,\omega)\cdot\pi_{\omega}}{\sum_{\omega:j\in\omega}\pi_{\omega}}. \tag{3}$$

In the causal attribution method, the incremental conversion rates for each publisher do not necessarily add up to the conversion rate, that is, $\sum_{j\in\omega}\Delta(j,\omega)\neq\hat{\rho}_{\omega}$.[15] This means the credit assigned to each publisher for each conversion does not add up to one, leading to an accounting problem. A proposed solution in the literature suggests using the Shapley values to address this issue, based on Shapley (1953). To calculate the contributions of publisher j in a sequence of touchpoints ω, it takes into account the incremental conversion rates for every subsequence[16]

$$\varphi(j,\omega) \equiv \sum_{\tilde{\omega}:j\in\tilde{\omega}\subseteq\omega}\frac{(|\tilde{\omega}|-1)!\,(|\omega|-|\tilde{\omega}|)!}{|\omega|!}\cdot\Delta(j,\tilde{\omega}).$$

Similar to a causal attribution method, the *Shapley-value attribution* method computes the attributed conversion rate replacing the incremental conversion rate with the Shapley values, that is,

$$r_j \equiv \mathbb{E}_T\left[\varphi(j,T)|j\in T\right] = \frac{\sum_{\omega:j\in\omega}\varphi(j,\omega)\cdot\pi_{\omega}}{\sum_{\omega:j\in\omega}\pi_{\omega}}. \tag{4}$$

I assume that when the advertiser is calculating the attributed conversion rate r_j for publisher j, the conversion rate r_{ω} for $j\notin\omega$ takes its stationary value.[17] This assumption

[14] These estimates may not be accurate, and I regard this approach only as a heuristic. The reason is that for $\Delta(j,\omega)$ to be unbiased, a standard assumption is selection-on-observable, that is, publisher j's display is independent of the potential conversion outcomes conditional on user observables. As shown in Section 4, this assumption is violated in my empirical application.

[15] For instance, for $\omega=(1,2)$, suppose that $\hat{\rho}_{(1,2)}=1$, $\hat{\rho}_{(1)}=0$, and $\hat{\rho}_{(2)}=0$, then $\Delta(1,\omega)=1$ and $\Delta(2,\omega)=1$, which do not sum up to $\hat{\rho}_{(1,2)}$.

[16] For instance, for $\omega=(1,2)$, $\varphi(1,\omega)=\frac{1}{2}(\rho_{(1,2)}-\rho_{(2)})+\frac{1}{2}(\rho_{(1)}-\rho_{\emptyset})$, and $\varphi(2,\omega)=\frac{1}{2}(\rho_{(1,2)}-\rho_{(1)})+\frac{1}{2}(\rho_{(2)}-\rho_{\emptyset})$.

[17] This assumption ensures consistency with the oblivious strategies, detailed in the next subsection. For instance, when calculating the attributed conversion rate for publisher 1, data-driven attribution methods may utilize the conversion rate $\hat{\rho}_{(2)}$, which does not involve publisher 1, and $\hat{\rho}_{(2)}$ is assumed to

makes r_j a function only of π_j in (4), where $\pi_j = (\pi_{\omega,y})_{j\in\omega}$ is the vector of the probability masses where publisher j is in the sequence of touchpoints.

Let us first summarize some common properties of the attributed conversion rate across standard attribution methods. In (2), (3), and (4), the attributed conversion rates for publisher j are all *homogeneous of degree* 0 in π_j. This property means that if we scale the probability masses within π_j by a constant factor,[18] the attributed conversion rate remains unchanged. Another interpretation is that the attributed conversion rates are determined by the *relative* proportions of the probability masses within π_j, rather than their *absolute* magnitudes. This property aligns intuitively with the definition of conversion rates, being a ratio, and I will adopt it as a requirement for all attribution methods studied in this paper.

Moreover, across all standard attribution methods, the attributed conversion rates r_j for publisher j satisfies $\frac{\partial r_j}{\partial \pi_{\omega,1}} \geq \frac{\partial r_j}{\partial \pi_{\omega,0}}$ for $j \in \omega$. This property means that compared with when there is not a conversion after publisher j's display, when a conversion happens, the credit given to publisher j is higher. I will also adopt this property as a requirement for all attribution methods studied in this paper.

In the two-step procedure of the attribution algorithm, the advertiser first computes the attributed conversion rates (r_j) for each publisher using one of the methods described. Subsequently, based on these rates, the advertiser computes the bid for the following period, denoted as b'_j for publisher j, according to

$$b'_j = \psi_j(b_j, r_j),$$

where $\psi_j : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$ is a smooth function that increases in r_j, and its relation to (1) is discussed below.[19] Furthermore, I impose a regularity condition that for any sequence of touchpoints ω and conversion indicator y, $\left| \frac{\partial \psi_j(b_j,r_j)}{\partial \pi_{\omega,y}} \cdot \frac{\sum_{\omega: j\in\omega} \pi_\omega}{\psi_j(b_j,r_j)} \right| \leq 1 - \frac{\partial \psi_j(b_j,r_j)}{\partial b_j}$, implying that the bids for the subsequent period are not excessively elastic to probabilities.

In summary, the *attribution algorithm* Ψ described in (1) can be formally defined as below. Note that this algorithm can also be regarded as an "attribution-based automated bidding algorithm," but I will refer to it as an attribution algorithm for brevity. Recall that Ψ is a vector-valued function, denote its j^{th} component as Ψ_j, then we can express

$$\Psi_j(b_j, \pi_j) \equiv \psi_j(b_j, r_j(\pi_j)).$$

In other words, the next period's bid for publisher j is determined in two steps: the algorithm first computes the attributed conversion rate r_j, which is a function of π_j.

take its stationary value.

[18] Recall that π_j is the subset of the probability masses where publisher j is in the sequence of touchpoints, so the sum over its elements can range from 0 to 1.

[19] See Appendix A.1 for a micro-foundation.

Then, the attributed conversion rate r_j is employed to adjust the bids b'_j for the following period. The first step of calculating the attributed conversion rate involves the specific attribution methods, and the second step of determining the bids involves the bidding rule.

2.4 Oblivious Equilibrium

Having described standard attribution algorithms, I now turn to characterizing the equilibrium they induce. In a dynamic Bayesian game, the strategy space is typically vast and the equilibrium strategy profile can be complex. To derive more insights, I focus on oblivious strategy profiles with stationary bid vectors, based on Weintraub et al. (2008).

Definition 2.1. We say a strategy is *oblivious* for publisher j if its advertisement display rule $a_j^{(t)}$ in period t solely depends on its respective private state variable, the bid $b_j^{(t)}$.

Note that the advertisement display rule $a_j^{(t)}$ is a function that takes user opportunity costs as inputs. This definition means that, under an oblivious strategy, whether a user i in period t is shown an advertisement by publisher j solely depends on the user's opportunity cost c_{ij} and the bid $b_j^{(t)}$.

An oblivious strategy is similar to a Markov strategy wherein player actions rely on the state variable. However, an oblivious strategy differs in that each player's action depends only on its private state variable. Specifically, an oblivious strategy implies a publisher's advertisement display rule relies on the bid the advertiser submits to it, but not the bid the advertiser submits to other publishers. This scenario is plausible in practice, as a publisher often does not observe the advertiser's bids for other publishers, making an oblivious strategy closer to reality.

Moreover, a publisher's equilibrium strategy is required to be "rational" with the following intuition. If the bid vector b^* "converges" in the long run, a publisher's belief about the bids submitted to the other players should be consistent with this bid vector b^*, and its belief about the other publishers' actions should be consistent with their actions under b^*. Denote the strategy profile of the publishers as $\mathbf{a} \equiv (\mathbf{a}_j)$, then this long-run stationary bid vector b^* is defined below.

Definition 2.2. We say a bid vector b is *stationary* under the publishers' oblivious strategy profile $\mathbf{a}$ and the advertiser's attribution algorithm Ψ, if b, $\mathbf{a}$ and Ψ satisfy the following conditions: if the advertiser sets $b^{(t)} = b$ in period t, the publishers act according to $\mathbf{a}$ given b, and then the advertiser sets $b^{(t+1)}$ based on Ψ, then $b^{(t+1)} = b$.

The concept of stationarity captures a state where the bid vector remains constant over time: if the bid vector is b in one period, it remains b in the subsequent period.[20]

[20] Typically, stationarity is defined in the literature using the probability distribution of the state variable. However, given the deterministic nature of the state variable transitions in this model, the stationary distribution becomes degenerate.

It is worth noting that the stationarity of a bid vector b is contingent on the publishers' strategy profile $\mathbf{a}$ and the advertiser's bidding rule Ψ.

Now I define the equilibrium of the game. I utilize the notation for the dynamic game as $G(\Psi)$ when the advertiser adjusts the bids according to the rule Ψ.[21]

Definition 2.3. An *oblivious equilibrium* of game $G(\Psi)$ is a pair of oblivious strategy profile $\mathbf{a}^*$ and stationary bid vector b^*, denoted as $(\mathbf{a}^*, b^*)$, which satisfies two conditions: (i) each publisher's strategy is the best response to the other publishers' strategies within all oblivious strategies such that each publisher's belief of other publishers' state variables and actions is consistent with b^*; (ii) the bid vector b^* is stationary under $\mathbf{a}^*$ and Ψ.

In each period, after the bids are determined by the advertiser, the publishers make advertisement display decisions. To characterize the equilibrium outcome, let us first examine the publishers' problems.

In any period, for publisher j, the stage payoff under a per-impression bid b_j and advertisement display rule a_j is

$$\mathbb{E}[a_j \cdot (b_j - c_j)] = \int a_{ij}(b_j - c_{ij})\,\mathrm{d}i.$$

In the oblivious strategy framework, the value function for publisher j is exclusively a function of its state variable, assuming the states and actions of other publishers are consistent with the stationary counterparts. Denote the value function for a publisher as $V_j(b_j)$, then the corresponding Bellman equation is

$$\begin{aligned} V_j(b_j) = \max_{a_j} \quad & \int a_{ij}(b_j - c_{ij})\,\mathrm{d}i + \delta \cdot V_j(b_j'), \\ \text{s.t.} \quad & \qquad b_j' = \psi_j(b_j, r_j). \end{aligned} \tag{5}$$

Here, r_j refers to the attributed conversion rate computed through the advertiser's attribution algorithm, given that other publishers' advertisement display decisions are consistent with the stationary decisions. Since r_j relies on the masses π_j of users with different touchpoints and conversion outcomes, and π_j depends on the advertisement display strategy a_j, r_j is a function of a_j. Besides, everything here is conditioned on publisher j's information, which is omitted to ease notation.

Denote the state-action value function

$$Q_j(b_j, a_j) \equiv \int a_{ij}(b_j - c_{ij})\,\mathrm{d}i + \delta \cdot V_j(b_j'),$$

then the first-order condition of the problem (5) implies that the best-response display

[21] In the notation, the initial state $b^{(1)}$, that is, the bids in the first period, is not explicitly stated, as I focus on the stationary state b^*, which may not directly be linked to the initial state.

strategy a_j^* satisfies

$$a_{ij}^* = \mathbb{1}\left(\frac{\partial Q_j}{\partial a_{ij}} \geq 0\right).$$

This equation suggests that when deciding on displaying an advertisement to user i, publisher j takes into account not only the immediate cost c_{ij} and benefit b_j but also the influence on future profits because of the changes in bids.[22] Hence, attribution algorithms introduce dynamic incentives for publishers, differing from the incentives in static pay-per-impression schemes. These incentives are captured by the state-action value function Q. It is also important to note that different attribution algorithms can generate different Q functions, leading to different equilibrium outcomes.

The main result of this section is that the incentives in the dynamic model can be replicated by a static contract-like incentive. To define this static incentive, one first needs to slightly extend the stage game to incorporate more sophisticated incentives. Recall that in the stage game, publisher j receives a payment b_j after displaying an advertisement. Now, the static model is extended to allow for incentive schemes that are contingent on the touchpoint sequence and the conversion outcome. That is, the incentive scheme g_j for publisher j takes the touchpoints and conversion as input, such that the advertiser pays $g_j(\omega, y)$ to publisher j when a user has touchpoint ω and conversion outcome y. The stage game presented in Section 2.1 can be seen as a special case such that $g_j(\omega, y) = b_j$ when $j \in \omega$. Let $g \equiv (g_j)$, and with a slight abuse of notation, denote this Bayesian game as $\Gamma(g)$. Then, one can define a Bayesian Nash equilibrium of this static game in a standard way.

The following proposition shows that the incentives in the dynamic environment can be replicated by the static contract-like incentive, with the same equilibrium outcomes and transfers.

Proposition 2.1. *Suppose Ψ is a bidding rule such that for each publisher j, Ψ_j is homogeneous of degree 0 in π_j. Let $(\boldsymbol{a}^*, b^*)$ denote an oblivious equilibrium of the dynamic game $G(\Psi)$, and $\boldsymbol{a}^*|_{b^*}$ denote the publishers' strategy profile when the vector of bids is b^*. Then there exists an incentive scheme $\tilde{g}$ such that $\tilde{g}_j(\omega, 1) \geq \tilde{g}_j(\omega, 0) \geq 0$ for any ω, and:*

1. *The strategy profile $\boldsymbol{a}^*|_{b^*}$ is a Bayesian Nash equilibrium of the static game $\Gamma(\tilde{g})$.*
2. *The equilibrium payoffs of the advertiser and the publishers under $\boldsymbol{a}^*|_{b^*}$ in the static game $\Gamma(\tilde{g})$ equal their equilibrium payoffs under $(\boldsymbol{a}^*, b^*)$ in the dynamic game $G(\Psi)$.*

The proof of Proposition 2.1 and other propositions are presented in Appendix C.

Note that the incentive scheme $\tilde{g}$ satisfies $\tilde{g}_j(\omega, 1) \geq \tilde{g}_j(\omega, 0) \geq 0$ for any ω such that $j \in \omega$, which means that it operates similarly to a contract: publisher j is compensated

[22] In practice, this dynamic consideration can be implemented by "bid modifications" on the publisher's side, as detailed in Appendix A.2. This discussion provides a micro-foundation for such bid modifications.

with $\tilde{g}_j(\omega, 0)$ if it displays the advertisement and the touchpoint sequence is ω, and it receives an additional compensation $\tilde{g}_j(\omega, 1) - \tilde{g}_j(\omega, 0)$ if this display leads to a conversion. It is worth mentioning that such contract-like incentives can be directly implemented through existing pay-per-action mechanisms. See Appendix A.3 for a detailed discussion.

Given that the incentives in the equilibrium of the dynamic game can be replicated by a static incentive scheme, I henceforth refer to the corresponding scheme in the static game as the *static equivalent* of the attribution algorithm in the dynamic game. This characterization also eases the computation of counterfactual outcomes under these dynamic incentives.

2.5 Static Equivalents

Upon establishing Proposition 2.1, we can now express the incentives created by rule-based and data-driven attribution algorithms based on (2), (3), and (4).

The static equivalents of the algorithms take a similar form, a mixture of a pay-per-impression scheme and another scheme. The mixture weight η_j for publisher j takes the form

$$\eta_j \equiv \delta \cdot \left(1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial r_j} \cdot \frac{r_j}{\psi_j(b_j^*, r_j)}, \tag{6}$$

which is the product of the discount factor, a multiplier that captures the duration the bids are carried over, and the elasticity of ψ_j with respect to the attributed conversion rate r_j.

For rule-based algorithm Ψ^{RB} with weight function $\chi(j, \omega)$ as defined by (2), its static equivalent is a mixture of a pay-per-impression scheme and a pay-per-attributed-conversion scheme.

Proposition 2.2. *Suppose $(\boldsymbol{a}^*, b^*)$ is an oblivious equilibrium of the dynamic game $G(\Psi^{RB})$. Its static equivalent, denoted as $\tilde{g}^{RB}$, is given by*

$$\tilde{g}_j^{RB}(\omega, y) = (1 - \eta_j) \cdot b_j^* + \eta_j \cdot v_j^* \cdot \chi(j, \omega) \cdot \mathbb{1}(y = 1), \tag{7}$$

which is a mixture of a pay-per-impression scheme and a pay-per-attributed-conversion scheme. The mixture weight η_j is given by (6), and $v_j^* \equiv \frac{b_j^*}{r_j^{RB}}$ is the effective payment per attributed conversion.

The underlying intuition of this proposition is that, if a user converts and is attributed to a particular publisher, their attributed conversion rate rises, leading to an increase in the advertiser's future bids. Conversely, if a user does not convert or does convert but is not attributed to this publisher, the attributed conversion rate falls and the advertiser's future bids will be correspondingly reduced. This algorithm creates stronger incentives for a publisher to display the advertisement to the users whose conversions are more likely to be attributed to this publisher.

For algorithm Ψ^{C} based on the causal attribution as defined by (3), its static equivalent is characterized as follows.

Proposition 2.3. *Suppose $(\boldsymbol{a}^*, b^*)$ is an oblivious equilibrium of the dynamic game $G(\Psi^C)$. Its static equivalent, denoted as $\tilde{g}^C$, is given by*

$$\tilde{g}_j^C(\omega, y) = (1 - \eta_j) \cdot b_j^* + \eta_j \cdot v_j^* \cdot (\mathbb{1}(y = 1) - \rho_{\omega \setminus \{j\}}), \tag{8}$$

which is a mixture of a pay-per-impression scheme and a scheme paying $v_j^* \cdot (\mathbb{1}(y = 1) - \rho_{\omega\setminus\{j\}})$ for touchpoint sequence ω and conversion indicator y. In this equation, $\rho_{\omega\setminus\{j\}}$ is the conversion rate of users with touchpoint sequence $\omega\setminus\{j\}$ in equilibrium. The mixture weight η_j is given by (6), and $v_j^* \equiv \frac{b_j^*}{r_j^C}$ is the effective payment per attributed conversion.

Let us compare the static equivalents of the rule-based attribution algorithm and the causal attribution algorithm. The pay-per-impression scheme components of the static equivalents, $(1 - \eta_j) \cdot b_j^*$, are similar. However, their respective second components are different. For the causal attribution algorithm, instead of a pay-per-attributed-conversion scheme, the publisher is subject to a v_j^* incentive upon user conversion, and a penalty of $v_j^* \cdot \rho_{\omega\setminus\{j\}}$ for touchpoint sequence ω. Note that $\rho_{\omega\setminus\{j\}}$ represents the counterfactual conversion rate when publisher j does not display the advertisement. It means that this algorithm incentivizes incremental conversions touchpoint sequences by giving a larger payment to conversions with a low counterfactual conversion rate $\rho_{\omega\setminus\{j\}}$.

For algorithm Ψ^{S} based on the Shapley attribution, as defined by (4), its static equivalent is characterized as follows.

Proposition 2.4. *Suppose $(\boldsymbol{a}^*, b^*)$ is an oblivious equilibrium of the dynamic game $G(\Psi^S)$. Its static equivalent, denoted as $\tilde{g}^S$, is given by*

$$\tilde{g}_j^S(\omega, y) = (1 - \eta_j) \cdot b_j^* + \eta_j \cdot v_j^* \cdot \left(\zeta_j(\omega) \cdot (\mathbb{1}(y = 1) - \rho_\omega) + \varphi(j, \omega)\right), \tag{9}$$

which is a mixture of a pay-per-impression scheme and scheme paying $v_j^* \cdot (\zeta_j(\omega) \cdot (\mathbb{1}(y = 1) - \rho_\omega) + \varphi(j, \omega))$ for touchpoint sequence ω and conversion indicator y. In this equation, ρ_ω is the conversion rate of users with touchpoint sequence ω in equilibrium, and $\varphi(j, \omega)$ is the corresponding Shapley value for publisher j in equilibrium. The mixture weight η_j is given by (6), $v_j^* \equiv \frac{b_j^*}{r_j^S}$ is the effective payment per attributed conversion, and $\zeta_j(\omega) \equiv \sum_{\tilde{\omega}:\omega\subseteq\tilde{\omega}} \frac{(|\omega|-1)!(|\tilde{\omega}|-|\omega|)!}{|\tilde{\omega}|!} \cdot \frac{\pi_{\tilde{\omega}}}{\pi_\omega}$ is a multiplier assigned to touchpoint sequence ω.

The pay-per-impression scheme component, $(1-\eta_j) \cdot b_j^*$, is similar to the corresponding components of the rule-based attribution algorithms and the causal attribution algorithm.

Its second component is more complicated. For the touchpoint sequence ω, the transfer is $v_j^* \cdot (-\zeta_j(\omega) \cdot \rho_\omega + \varphi(j, \omega))$, with an additional $v_j^* \cdot \zeta_j(\omega)$ incentive if a conversion occurs. The definition of $\zeta_j(\omega)$ implies that $\zeta_j(\omega)$ decreases in ω, that is, for $j \in \omega_1 \subseteq \omega_2$, $\zeta_j(\omega_1) \geq \zeta_j(\omega_2)$. Thus, similar to a linear attribution algorithm, this scheme assigns lower incentives to touchpoint sequences with a larger number of publishers.

To gain more intuition, some special cases are provided in Appendix D.

3 Optimal Incentives

With a variety of attribution algorithms, an advertiser could be interested in which attribution algorithm could help address the issue of information asymmetry and lead to the highest profit. As Section 2 shows that the incentives in the dynamic game's equilibrium can be replicated by static incentive schemes, it amounts to optimizing the advertiser payoff over the incentive schemes in the static model. To achieve this goal, in this section, I further characterize the equilibrium of static game $\Gamma(g)$ and optimize over the possible incentive scheme g. Moreover, it is worth mentioning that the static incentive schemes considered in this section can also be directly implemented through existing pay-per-action mechanisms,[23] so the static incentives *per se* is also worth investigation.

Note that the model considered in this section has a single focal advertiser. One may be interested in whether the results apply to the extension in Appendix B.1, where there are multiple advertisers and multiple publishers. The analyses here could be interpreted as a best-response strategy analysis in the extension regarding the focal advertiser, holding constant the actions of other advertisers. See Appendix B.1 for a detailed discussion.

3.1 Bayesian Nash Equilibrium

The setup of the static game is the same as the stage game in Section 2.1, except that the incentive schemes are more flexible. Specifically, the incentive scheme $g_j : \mathcal{T} \times \{0, 1\} \to \mathbb{R}_+$ for publisher j takes the touchpoints and conversion as input, such that the advertiser pays $g_j(\omega, y)$ to publisher j when a user has touchpoint ω and conversion outcome y. Here, $\mathcal{T}$ is the set of potential touchpoint sequences.

Once the advertiser sets up the incentive scheme $g = (g_j(\cdot))$, the publishers participate in a Bayesian game. Recall that the game is denoted as $\Gamma(g)$, and the strategy profile of the publishers is denoted as $\mathbf{a} = (\mathbf{a}_j)$. Each publisher j only observes their respective opportunity cost c_j, with other publishers' opportunity costs inferred in a Bayesian manner. The solution concept is thus the Bayesian Nash equilibrium.

To investigate the Bayesian Nash equilibrium, the initial step is to examine each publisher's best response to other publishers' behavior. The user subscript i is dropped

[23] See Appendix A.3 for a detailed discussion.

hereafter to ease notation unless stated otherwise. Given the opportunity cost c_j and the strategies $\mathbf{a}_{-j}$ of other publishers, which are functions of the opportunity cost vector $\mathbf{c}_{-j}$, publisher j's best response is to display the advertisement if and only if the expected revenue $R_j(c_j;\mathbf{a}_{-j})$ is no less than the opportunity cost c_j:

$$a_j^*(c_j;\mathbf{a}_{-j}) = \mathbb{1}\left(R_j(c_j;\mathbf{a}_{-j}) \geq c_j\right).$$

To determine $R_j(c_j;\mathbf{a}_{-j})$, we need to consider that the actions of other publishers are contingent on their opportunity costs. Thus, we should integrate over c_{-j} given c_j:

$$R_j(c_j;\mathbf{a}_{-j}) \equiv \mathbb{E}_{c_{-j},Y|c_j}\left[g_j(T,Y)|c_j\right] = \mathbb{E}_{c_{-j}|c_j}\left[g_j(T,1)\cdot h(T,c) + g_j(T,0)\cdot(1-h(T,c))|c_j\right],$$

where touchpoint sequence T is determined jointly by the browsing path and the advertisement display decisions, and the distribution of Y can be characterized by the conversion rate function

$$h(\omega,c) \equiv \mathbb{P}(Y=1|T=\omega,c).$$

In this equation, the conversion rate can potentially be a function of c in a Heckman (1979) selection model fashion. That is, the opportunity costs c are utilized by the publishers to select whether an advertisement is shown, and the opportunity costs can be correlated with the conversion outcomes.[24] I assume the distributions are sufficiently smooth enough so that $h(\omega,c)$ is smooth in c for a fixed ω and $R_j(c_j;\mathbf{a}_{-j})$ is smooth in c_j given $\mathbf{a}_{-j}$ for any publisher j. I also impose a regularity condition $0 < h(\omega,c) < 1$ for any ω and c.

Based on the best response analysis, we can now formally define the solution to the static game. A strategy profile $\mathbf{a}^*$ of the game $\Gamma(g)$ is a Bayesian Nash equilibrium if for each publisher j, its strategy a_j^* is the best response to the strategies $\mathbf{a}_{-j}^*$ of other publishers.

In general, publisher j's strategy a_j may not be monotone and could be complicated. I make an additional simplifying assumption that the publisher strategy profile consists of *thresholding strategies*, that is, for each publisher j, there exists a $\bar{c}_j$ such that

$$a_j^*(c_j) = \mathbb{1}(c_j \leq \bar{c}_j).$$

Employing a thresholding strategy means displaying the advertisement to users with comparatively low opportunity costs. For a given $\mathbf{a}_{-j}$, the corresponding threshold $\bar{c}_j$,

[24] This dependency can be micro-founded by the following. The user types include two parts, the opportunity cost vector c and an unobserved variable ξ, which can be correlated with c. The conversion rate is a function of ξ. However, since ξ is unobserved, one can only take expectations conditional on c and identify the function $h(\omega,c)$.

which I also refer to as the *marginal user type*, satisfies

$$R_j(\bar{c}_j; \mathbf{a}_{-j}) = \bar{c}_j.$$

Appendix E presents a set of sufficient conditions on the primitives to ensure that any Bayesian Nash equilibrium is a thresholding strategy profile. Moreover, recall that everything here is conditional on user browsing paths, so for users with different browsing paths, the corresponding thresholds $\bar{c}_j$ could be different.

3.2 Advertiser's Incentive Design Problem

Now I turn to the advertiser's problem. The advertiser faces an incentive design problem, effectively determining the incentive scheme g that leads to the highest profit.

Formally, suppose that each conversion results in a conversion value of K. Denote the possible set of parameters as $\mathcal{G}$, which is detailed shortly. Moreover, denote the Bayesian Nash equilibria of the game as $\mathrm{BNE}(\Gamma(g))$. The advertiser's objective is to maximize the conversion value net of the compensations.

$$\begin{array}{cc} \max\limits_{g \in \mathcal{G}} & K \cdot \mathbb{E}_{c,Y}[Y(\mathbf{a}^*)] - \sum_{j=1}^{J} \mathbb{E}_{c,Y}\left[g_j(T(\mathbf{a}^*), Y(\mathbf{a}^*))\right] \\ \text{s.t.} & \mathbf{a}^* \in \mathrm{BNE}(\Gamma(g)) \end{array} \tag{10}$$

The expectation is taken over both the opportunity cost vector c and the conversion event Y, since the publishers' strategies depend on the opportunity cost vector, and the conversion event is stochastic given the display decisions.

The objective function consists of two parts. In the first part, $\mathbb{E}_{c,Y}[Y(\mathbf{a}^*)]$ is the expected number of conversions. In the second term, $\mathbb{E}_{c,Y}\left[g_j(T(\mathbf{a}^*), Y(\mathbf{a}^*))\right]$ is the expected compensation to publisher j. The distributions of touchpoints T and conversion Y depend on the publishers' strategy profile $\mathbf{a}^*$, where $\mathbf{a}^*$ is a Bayesian Nash equilibrium of the game $\Gamma(g)$.

Recall that one reason for considering the optimization problem (10) is the following. We would like to optimize the advertiser's profit over all attribution algorithms. Section 2 shows that an attribution algorithm can be replicated by a static incentive scheme. Let $\mathcal{G}$ be the set of static incentive schemes such that they are the static equivalents of some attribution algorithms, then we can optimize over incentive schemes in $\mathcal{G}$. It means that one way to solve (10) is to characterize this set $\mathcal{G}$ and solve the optimization problem under the constraint $g \in \mathcal{G}$. Although feasible, this procedure is tedious and does not add much insight. Instead, I will first solve (10) with a relaxed constraint. I then show that for the optimal incentive g^*, there exists an attribution algorithm whose static equivalent is g^* (see Appendix F). Then, g^* has to be the optimal solution for (10) as well.

This relaxed constraint is that the compensations are nonnegative and have an upper bound M, that is, there is an upper bound M such that $0 \leq g_j(\omega, 0) \leq g_j(\omega, 1) \leq M$ for all publisher j and touchpoint ω. Moreover, M is set high enough such that any compensation exceeding M would be considered unprofitable. One can show that $\mathcal{G}$ is a subset of the set $\mathcal{G}$: Proposition 2.1 has shown that the static equivalent satisfies $0 \leq g_j(\omega, 0) \leq g_j(\omega, 1)$, and the upper bound can be ensured by regularity conditions on the attribution algorithms.

Moreover, in the counterfactual analysis in Section 5, some constrained variants of the optimization problem (10) will be solved and serve as benchmarks. In other words, the advertiser's optimization of g is constrained to a subset. Some standard classes of schemes are as follows:

The class of *pay-per-impression* schemes encompass g such that for $j \in \omega$,

$$g_j^{\mathrm{I}}(\omega, y) = b_j, \tag{11}$$

meaning that the advertiser pays b_j to publisher j if j displays the advertisement.

The class of *pay-per-conversion* schemes encompass g such that for $j \in \omega$,

$$g_j^{\mathrm{C}}(\omega, y) = v_j \cdot \mathbb{1}(y = 1), \tag{12}$$

meaning that the advertiser pays v_j to publisher j if j displays the advertisement and the user converts.

3.3 Characterization of Optimal Scheme

The advertiser's problem is to design the incentive scheme g in the optimization problem (10). To characterize the optimal incentive scheme, I first define some properties that these schemes may have.

Definition 3.1 (Competitive Scheme)**.** Suppose g is an incentive scheme. We say g is *competitive* if for every publisher j, touchpoint sequence ω, and conversion indicator y, it follows that $g_j(\omega, y) = 0$ if $\omega \neq (j)$.

A competitive scheme fosters competition by ensuring that publisher j gets compensated only if j is the sole publisher within the sequence of touchpoints. For instance, if there are two publishers 1 and 2, and when both publisher 1 and publisher 2 display the advertisement, neither publisher 1 nor publisher 2 receives any compensation, then the scheme is competitive. However, publisher 1 receives a positive amount when only publisher 1 displays the advertisement, and similarly does publisher 2. This incentive scheme is similar to a tournament, in which only the best performer, in this case, the sole publisher that displays the advertisement, receives a payment (e.g., Lazear and Rosen, 1981; Green and Stokey, 1983; Nalebuff and Stiglitz, 1983). Note that in a competitive scheme, a publisher can be compensated even when the user does not convert. In other words, it does not require that a publisher be compensated if there is a conversion.

Another property is whether a publisher is compensated if the advertisement is displayed but results in no conversions. In a per-conversion scheme defined below, publisher j is compensated only upon conversion.

Definition 3.2 (Per-Conversion Scheme)**.** Suppose g is an incentive scheme. We say g is *per-conversion* if for every publisher j and touchpoint sequence ω, $g_j(\omega, 0) = 0$.

The characteristics of the optimal scheme hinge upon the primitives. An important primitive is the properties of joint distribution $f(c)$ of the opportunity costs.

Definition 3.3 (Affiliated Distribution)**.** Suppose f is the probability density function of a continuous random vector. We say f is *affiliated* if $\log f$ is super-modular.

Broadly, affiliation implies that a higher opportunity cost of a publisher makes a higher opportunity cost of other publishers more likely (Milgrom and Weber, 1982).

We first consider a simple case. When $h(\omega, c) \equiv h(\omega)$ is constant for a fixed ω, that is, there are no selection-on-unobservables issues, the following proposition characterizes the properties of the optimal scheme.

Proposition 3.1. *Suppose $h(\omega, c) \equiv h(\omega)$ is a constant for a fixed ω. If the opportunity cost distribution f exhibits affiliation, then the optimal static scheme g^* is competitive.*

The intuition is that, when opportunity costs are affiliated, if a publisher's opportunity cost is low, other publishers' opportunity costs are also likely to be low, so it is more likely that multiple publishers display the advertisement to a user. In other words, when multiple publishers display the advertisement to a user, it indicates that the opportunity costs are likely to be low, and thus, the advertiser need not pay a high amount to either publisher for the display. A competitive scheme can then reduce the advertiser's expenditure on users with low opportunity costs. Furthermore, a competitive scheme can create an additional incentive for a publisher to display to a user with a relatively high opportunity cost, as the opportunity costs for other publishers are likely to be high for this user, and this display is more likely to lead to a touchpoint sequence that only involves this publisher.

It is worth mentioning that the characteristics of the optimal scheme hinge upon the primitives. Utilizing the same proof techniques, one can show the following proposition: suppose $h(\omega, c) \equiv h(\omega)$ is a constant for a fixed ω. If $\log f$ is sub-modular, then the optimal static scheme g^* satisfies $g_j(\omega, y) = 0$ if ω does not involve all publishers. Note that the condition that $\log f$ is sub-modular is the opposite of affiliation. It implies that a higher opportunity cost of a publisher makes a lower opportunity cost of other publishers more likely. In this case, the form of the optimal scheme is also the opposite of a competitive scheme. It compensates publisher j only if the touchpoint sequence involves all publishers. This scheme can create an additional incentive for a publisher to display to a user with a relatively high opportunity cost, as the opportunity costs for other publishers are likely to be low for this user, and this display is more likely to lead to a touchpoint sequence that involves all publishers.

We now turn to a more general case when the conversion rate can be correlated with the opportunity costs.

Proposition 3.2. *Suppose the opportunity cost distribution f exhibits affiliation. If $h(\omega, c)$ is nondecreasing in c for any fixed ω and $\log h(\omega, c)$ is super-modular in $(\omega, -c)$, then optimal scheme g^* is both competitive and per-conversion.*

Compared to Proposition 3.1, Proposition 3.2 imposes two additional conditions: (i) $h(\omega, c)$ is nondecreasing in c for any fixed ω, and (ii) $\log h(\omega, c)$ is super-modular in $(\omega, -c)$. The two conditions serve different purposes:

Condition (i) leads to the result that the optimal scheme is per-conversion. Intuitively, condition (i) means that a higher opportunity cost is correlated with a higher conversion rate for this user. Thus, a per-conversion scheme can create an additional incentive for a publisher to display to a user with a relatively high opportunity cost, and this display is more likely to lead to a conversion.

Condition (ii) ensures that the intuition of Proposition 3.1 holds. Specifically, this condition implies that the ratio of conversion rates $\frac{h(\omega_0,c)}{h(\omega,c)}$ for $\omega_0 \equiv (j) \subset \omega$ is larger for users with higher opportunity costs. A higher ratio $\frac{h(\omega_0,c)}{h(\omega,c)}$ means that publisher j's display has a relatively large incremental effect compared with other publishers' display,[25] and in this case, it could be more cost-effective to compensate publisher j only when publisher j is the sole publisher before a conversion. A competitive per-conversion scheme then creates an additional incentive for a publisher to display to a user with a relatively high opportunity cost, which is correlated with both a higher probability of this display resulting in a touchpoint sequence that involves this publisher and a relatively large incremental effect of this publisher compared with other publishers' displays.

Moreover, utilizing the same proof techniques, one can show the following proposition: suppose the opportunity cost distribution f exhibits affiliation. If $h(\omega, c)$ is nonincreasing in c for any fixed ω, then g^* is competitive, and for every publisher j and touchpoint sequence ω, $g_j(\omega, 1) = g_j(\omega, 0)$ holds. It implies that under the optimal scheme, publisher j is compensated solely when only publisher j displays the advertisement, and publisher j receives no additional compensation if a conversion happens. Intuitively, in this case, an additional compensation after conversion is ineffective in incentivizing a publisher to display to a user with a relatively high opportunity cost, as a higher opportunity cost is

[25] Another interpretation of a higher ratio $\frac{h(\omega_0,c)}{h(\omega,c)}$ is that the publishers' displays are less complementary.

correlated with a lower conversion rate for this user. Thus, the optimal scheme is per-impression. This additional result is related to the literature on per-impression versus per-conversion schemes in online advertising. It provides a set of conditions when each dominates the other in the multi-publisher setting.

To summarize this section, Propositions 3.1 and 3.2 showcase how touchpoint sequence can be used to manage incentives. The touchpoint sequence gives advertisers additional information about user primitives so that advertisers can leverage this information to reduce the exposure to moral hazard.

4 Data and Estimation

The theoretical insights in Sections 2 and 3 prompt an empirical investigation. Specifically, the results reveal that the characteristics of the model primitives play a crucial role in determining the form of optimal incentives, so it is important to examine these characteristics from data. Moreover, an empirical analysis can quantify the effectiveness of the optimal incentives and standard attribution algorithms.

Because of the limitation of observational methods in online advertising, I partner with an advertiser to conduct an online advertising experiment. I first discuss the experiment design and then provide descriptive analyses. Subsequently, I describe the model parameterization and proceed to estimation.

4.1 Experiment Design

The experiment is carried out on $J = 2$ major publishers, indexed by 1 and 2, through an anonymous advertising exchange. The anonymous advertiser is an online service provider, which relies on online advertising for customer acquisition. The advertising campaign spans a 7-day period in 2023. The conversion value K is reported by the advertiser based on historical estimates. For the sake of desensitization, I conduct a linear transformation on the conversion value, bids, and opportunity costs, such that $K = 10,000$.

In the experiment, I participate in display advertising with randomized per-impression bids.[26] The bidding is governed by the following filtering conditions: (i) users are restricted to a major segment targeted by the advertiser, and (ii) I limit my bids to the first visits made by each user to either publisher. This condition is to avoid excessively long or complex touchpoint sequences.

For each impression meeting the filtering criteria, a bid is drawn from a distribution detailed shortly and is sent to the exchange. Winning an auction will result in the display of the advertisement and receipt of "bid feedback" from the advertising exchange. This

[26] In this experimental setup, I effectively act as a demand-side platform (DSP), which submits bids on behalf of advertisers in online advertising auctions.

feedback informs me of the lowest possible bid that could have still won the auction, which is effectively the publisher's opportunity cost in the model. Losing an auction results in no display of the advertisement and no receipt of bid feedback.

The distribution that the bids are drawn from is as follows. With probability $\frac{1}{2}$, a bid of $\bar{b} = 28$ is drawn. The bid is high in that the advertiser deems it unprofitable to acquire any users with $b > \bar{b}$.[27] Moreover, this high bid almost guarantees an auction win,[28] thereby allowing me to observe the advertisement opportunity cost of the publisher. With probability $\frac{1}{2}$, the bid is drawn uniformly from the range 0 to $\bar{b}$. This design results in a positive probability of losing the auction, introducing variations in touchpoint sequences to identify the conversion rate function h.

After an auction, the user's subsequent actions are tracked, including visits to the advertiser's site and purchases. I define a conversion action as a site visit because of the scant number of purchases in the data.

I observe only a handful of additional covariates other than the ones I use to filter users. The set of covariates includes the time of the impression, masked IP address, and device information. I conduct robustness checks incorporating fixed effects based on these additional covariates, but the fit does not show considerable improvement. Therefore, in the main specification, no covariates are included.

4.2 Descriptive Analysis

The data consist of $N = 117,905$ multi-homing users.[29] Of these users, 56% initiate their browsing with publisher 1, resulting in a browsing path $(1, 2)$. The remaining users initiate their browsing with publisher 2, resulting in a browsing path $(2, 1)$.

Figure 1 presents the marginal and joint density functions of opportunity costs for both publishers. The figure includes only the users for whom the highest bid, $\bar{b}$, was made on both publishers and the auctions were won. The axes are log-transformed for rescaling.

In Figure 1, the top left and bottom right panels represent the smoothed marginal density function of the opportunity costs for each publisher. The distributions are mildly left-skewed and exhibit two modes for each, one for low opportunity costs and one for high opportunity costs.

The bottom left panel portrays a contour plot of the joint density function of the opportunity costs. The correlation coefficient between the logarithm of the opportunity costs is approximately 0.24, suggesting a weak-to-medium positive correlation. This result

[27] The value of $\bar{b}$ is the product of the conversion value and the historical conversion rate, implying that bidding $\bar{b}$ is estimated to be break-even for the firm.

[28] Only around 0.3% of such auctions are lost, and I discard such data points.

[29] Users who browse a single publisher have been excluded, as the attribution problem only arises for multi-homing users. The estimation results and counterfactual ROI should be interpreted as the corresponding quantities conditional on multi-homing users.

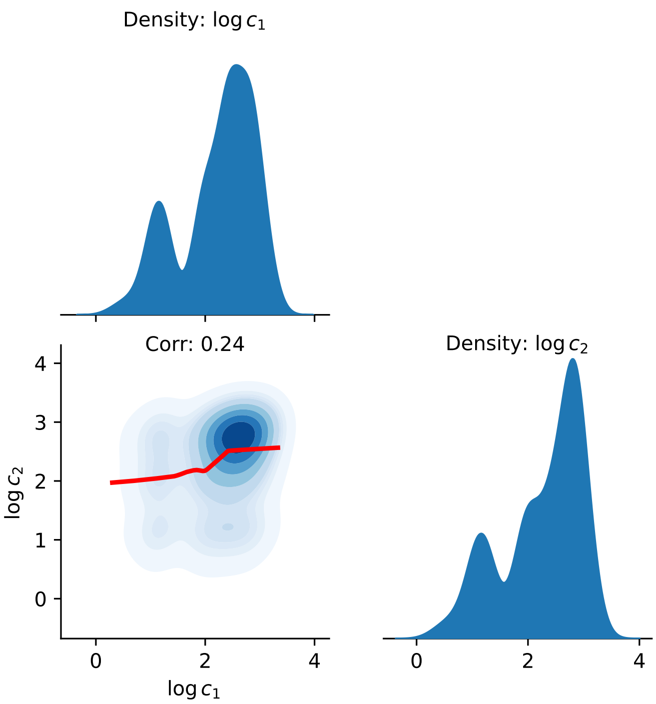

Figure 1: Marginal and joint density functions of opportunity costs

Note: This figure illustrates the marginal and joint density functions of log-transformed opportunity costs for both publishers, using data from users where the highest bid was placed on both publishers. The top-left and bottom-right panels show marginal density functions for $\log c_1$ and $\log c_2$. The bottom-left panel displays the joint density function, with a nonparametric regression line (in red) showing the relationship between $\log c_1$ and $\log c_2$. The correlation coefficient between $\log c_1$ and $\log c_2$ is 0.24, indicating a weak-to-medium positive correlation.

implies that Proposition 3.2 could potentially be applicable in this context. The red line indicates a nonparametric fit of $\log c_2$ against $\log c_1$. The fitted line is roughly linear but also shows some variation in areas with fewer data points.

To interpret the positive correlation of the opportunity costs, recall that opportunity costs can be influenced by other advertisers' bids and users' satisfaction levels with the advertisement. First, other advertisers may acquire data from data vendors, allowing them to identify high-value users and place high bids for them, leading to correlated bids across publishers. Second, user behavior across different publishers may be consistent. If users are averse to advertisements and refrain from engaging with them on one publisher, they are likely to do the same with other publishers. Both reasons could contribute to the opportunity cost, resulting in a positive correlation.

Touchpoints	N	Conversion rate
$(1, 2)$	40,753	0.589%
(1)	10,995	0.209%
(2)	10,968	0.192%
$\emptyset$	3,193	0.000%
Total	65,909	0.431%

(a) Users with browsing path $(1, 2)$

Touchpoints	N	Conversion rate
$(2, 1)$	31,619	0.601%
(1)	9,490	0.316%
(2)	8,340	0.269%
$\emptyset$	2,547	0.000%
Total	51,996	0.465%

(b) Users with browsing path $(2, 1)$

Table 1: Conversion rates by touchpoint sequence

Note: This table provides summary statistics of conversion rates by user browsing path and touchpoint sequence. Touchpoint sequences are coded as follows: $(1, 2)$ and $(2, 1)$ indicate the user saw ads on both publishers, (1) or (2) indicates the user saw an ad on only publisher 1 or 2, and $\emptyset$ indicates the user saw no ads. Panel (a) includes the users with browsing path $(1, 2)$, meaning they visited publisher 1 and then publisher 2. Panel (b) includes the users with browsing path $(2, 1)$, meaning they visited publisher 2 and then publisher 1. The last row in each panel gives the statistics for all users with that browsing path.

Table 1 provides summary statistics of conversion rates. Panel (a) restricts to the users with browsing path $(1, 2)$, and Panel (b) restricts to the users with browsing path $(2, 1)$. A majority of the bids are high, resulting in frequent auction wins and touchpoint sequences with both publishers. For users not shown advertisements by either publisher, no site visits to the advertiser are observed.

Based on the information in Table 1, the advertisements on both publishers exhibit a complementary effect. Exposure to advertisements from both publishers leads to a conversion rate exceeding the sum of conversion rates from exposure to either publisher. Additionally, the complementarity is stronger for the users with browsing path $(1, 2)$ compared to the users with browsing path $(2, 1)$.

4.3 Parameterization

I now describe a parsimonious parameterization for the structural model, to which I will subsequently fit the collected data. The experiment does not involve any dynamic bid adjustments, and no dynamic incentives are introduced, which ensures a clean identification of the primitive parameters. Specifically, it reduces to a static model, and an auction is won if and only if my per-impression bid is no less than the opportunity cost.

The dataset can be denoted as $\{(b_i, c_i, T_i, Y_i)\}$,[30] where $b_i = (b_{ij})$ represents the vector of submitted bids, $c_i = (c_{ij})$ refers to the vector of publisher opportunity costs, which is "truncated" in the following sense. When the auction for user i on publisher j is won, that is, $c_{ij} \leq b_{ij}$, the exact value of c_{ij} is observed; however, if the auction is lost, the exact value of c_{ij} is not reported and one can only infer that $c_{ij} > b_{ij}$.

I assume the logarithm of the opportunity costs admits a normal distribution. For publisher j,

$$\log c_{ij} = \gamma_j + \epsilon_{ij},$$

where the error terms

$$\epsilon_i = \begin{pmatrix} \epsilon_{i1} \\ \epsilon_{i2} \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \underbrace{\begin{pmatrix} \sigma_1^2 & \varrho\sigma_1\sigma_2 \\ \varrho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}}_{=\Sigma} \right).$$

In this specification, the correlation coefficient ϱ captures the correlation of opportunity costs across publishers parsimoniously.

As suggested in Section 3, conversion rates could be different conditional on users with different opportunity costs. To model this dependency, I take a selection model approach as outlined in Heckman (1979). The conversion rate for a user with touchpoint sequence ω and opportunity cost vector c is modeled as

$$\Phi^{-1}(h(\omega, c)) = \alpha_\omega + \sum_{j=1}^{J} \lambda_j \cdot \epsilon_j = \alpha_\omega + \sum_{j=1}^{J} \lambda_j \cdot (\log c_j - \gamma_j),$$

where Φ is the cumulative distribution function of a standard normal variable.[31] In this equation, the intercept α varies across different touchpoint sequences ω, while λ captures the dependency of the conversion rate on the opportunity costs.

Regarding the estimation procedure, I utilize the Maximum Likelihood Estimation (MLE) approach. Denoting the parameters as $\Theta = (\gamma, \Sigma, \alpha, \lambda)$, the likelihood function

[30] Recall that everything is conditional on the user browsing path H_i, and I omit it to simplify notation.

[31] This formulation is a probit model in essence. In a probit model, the standard normality assumption is essential for addressing the identification problem. Specifically, without this assumption, the location and scale parameters of the normal variable become unidentifiable.

can be expressed as follows:

In cases where both publishers display the advertisement to user i, hence c_i is observed, the likelihood is given as

$$\ell_i(\Theta) = \phi(\log c_i - \gamma, \Sigma) \cdot \left(\rho_i^{Y_i} \cdot (1-\rho_i)^{1-Y_i}\right),$$

where $\phi(x, \Sigma)$ is the density function of a multivariate normal random vector with mean 0 and covariance Σ, evaluated at x, and $\rho_i = h(T_i, c_i; \Theta)$ represents the likelihood of conversion given T_i and c_i.

If only publisher j but not publisher $-j$ displays the advertisement to user i, we observe c_{ij} but not $c_{i,-j}$, so one has to integrate $c_{i,-j}$ out to obtain the likelihood. Recall that the bids are on a per-impression basis, so an advertisement is displayed by publisher j if and only if $b_{ij} \geq c_{ij}$. Thus, we need to integrate $c_{i,-j}$ over where $c_{i,-j} > b_{i,-j}$, yielding

$$\ell_i(\Theta) = \int_{c_{i,-j} > b_{i,-j}} \phi(\log c_i - \gamma, \Sigma) \cdot \left(\rho_i^{Y_i} \cdot (1-\rho_i)^{1-Y_i}\right) \, \mathrm{d}c_{i,-j}.$$

In instances where neither publisher displays the advertisement to user i, the vector c_i remains unobserved. As in the previous scenario, we need to integrate the vector c_i over where $c_i \geq b_i$, leading to

$$\ell_i(\Theta) = \int_{c_i \geq b_i} \phi(\log c_i - \gamma, \Sigma) \cdot \left(\rho_i^{Y_i} \cdot (1-\rho_i)^{1-Y_i}\right) \mathrm{d}c_i = \int_{c_i \geq b_i} \phi(\log c_i - \gamma, \Sigma) \, \mathrm{d}c_i.$$

The last equality holds since $\rho_i = 0$ in this case, considering that there is no conversion observed where neither publisher displays the advertisement. Then, I proceed with standard MLE methodologies to derive the estimates and their standard errors.

4.4 Estimation Results

Table 2 presents the parameter estimates. For users with different browsing paths, namely $(1, 2)$ and $(2, 1)$, the parameters are estimated separately. For the users with both browsing paths, we see that the correlation coefficient ϱ between the logarithms of the opportunity costs are both statistically significant from 0, implying the opportunity cost vector is affiliated. This result indicates that Proposition 3.2 may apply to the estimated model. To invoke the proposition, it remains to be validated that the shape of the conversion rate function h satisfies the condition in Proposition 3.2.

Recall that the coefficient λ captures the potential dependency of conversion rates on the opportunity costs. A statistically significant λ indicates selections on unobservables, that is, the publishers use the opportunity costs c to select which users to display the advertisement to, and the opportunity costs are correlated with the users' subsequent conversion actions.

	Browsing path	
Parameter	(1, 2)	(2, 1)
γ_1	2.352^{***}	2.306^{***}
	(0.003)	(0.003)
γ_2	2.332^{***}	2.428^{***}
	(0.003)	(0.003)
σ_1	0.667^{***}	0.687^{***}
	(0.002)	(0.002)
σ_2	0.711^{***}	0.691^{***}
	(0.002)	(0.002)
ϱ	0.240^{***}	0.230^{***}
	(0.004)	(0.005)
$\alpha_{(1,2)}$	-2.511^{***}	/
	(0.023)	
$\alpha_{(2,1)}$	/	-2.508^{***}
		(0.029)
α_1	-2.882^{***}	-2.754^{***}
	(0.067)	(0.063)
α_2	-2.911^{***}	-2.779^{***}
	(0.070)	(0.070)
λ_1	0.054	-0.019
	(0.035)	(0.035)
λ_2	0.043	0.068^{*}
	(0.032)	(0.037)
N	65,909	51,996

Table 2: Parameter estimates

Note: This table presents parameter estimates from the structural model. Standard errors are in parentheses. The model is estimated separately for users with different browsing paths. The first column displays the estimates for the users with browsing path $(1,2)$, and the second column displays the estimates for the users with browsing path $(2,1)$. Parameters are defined as follows: γ_j is the mean of log opportunity costs for publisher j; σ_j is the standard deviation of log opportunity costs for publisher j; ϱ is the correlation coefficient between log opportunity costs; α_ω is the intercept for conversion rate function for touchpoint sequence ω; λ_j captures the dependency of conversion rate on opportunity costs for publisher j. $^{*}p < 0.1$; $^{**}p < 0.05$; $^{***}p < 0.01$.

For users with browsing path $(1, 2)$, neither λ_1 nor λ_2 differ significantly from zero. However, for clients with browsing path (2, 1), λ_2 is significantly and positively different from zero, suggesting a selection-on-unobservables problem, implying that the users with a higher opportunity cost at publisher 2 have higher conversion rates. In both cases, the conditions of Proposition 3.2 are fulfilled, indicating the optimal scheme is competitive.

5 Counterfactual Analysis

In the counterfactual analysis, I utilize the parameter estimates obtained in Section 4, assuming them as true values, and simulate the ensuing profits. Various incentive schemes are considered, as outlined:

1. Pay-per-impression defined by (11). I optimize over the set of parameters (b_j).
2. Pay-per-conversion defined by (12). I optimize over the set of parameters (v_j).
3. Last-touch attribution defined by (7), with $\chi^{\mathrm{LT}}(j, \omega) = \mathbb{1}(j = \mathrm{last}(\omega))$. There are two sets of free parameters $((1-\eta_j) \cdot b_j^*)$ and $(\eta_j \cdot v_j^*)$, and I optimize over all such possible parameters.
4. First-touch attribution defined by (7), with $\chi^{\mathrm{FT}}(j, \omega) = \mathbb{1}(j = \mathrm{first}(\omega))$. I optimize over the set of parameters $((1-\eta_j) \cdot b_j^*)$ and $(\eta_j \cdot v_j^*)$.
5. Linear attribution defined by (7), with $\chi^{\mathrm{L}}(j, \omega) = \dfrac{1}{|\omega|}$. I optimize over the set of parameters $((1-\eta_j) \cdot b_j^*)$ and $(\eta_j \cdot v_j^*)$.
6. Causal attribution defined by (8). I optimize over the set of parameters $((1-\eta_j) \cdot b_j^*)$, $(\eta_j \cdot v_j^*)$, and (ρ_ω), with the constraints that conversion rates (ρ_ω) are consistent with the corresponding conversion rates in the equilibrium.
7. Shapley attribution defined by (9). I optimize over the set of parameters $((1-\eta_j) \cdot b_j^*)$, $(\eta_j \cdot v_j^*)$, (ρ_ω) and $(\varphi(j, \omega))$, with the constraints that conversion rates (ρ_ω) and Shapley values $(\varphi(j, \omega))$ are consistent with the corresponding quantities in the equilibrium.
8. Optimal strategy. As proposed in Proposition 3.2, the optimal incentive scheme is per-conversion and competitive, so the free parameters are $(g_j(j, 1))$ and I optimize over them.

In practice, advertisers typically operate under a fixed advertising budget. Thus, I impose an upper bound on the budget in the stationary state.[32] As such, the optimization

[32] This constraint does not mean that the advertising expenditure is fixed. Instead, the advertising expenditure can vary over time when the publishers do not take the equilibrium actions.

problem (10) is resolved with a budget constraint

$$\sum_{j=1}^{J} \mathbb{E}\left[g_j(T(\mathbf{a}^*), Y(\mathbf{a}^*))\right] \leq B,$$

where $B \approx 16.31$ is derived from the advertiser's budget. This budget constraint allows a direct comparison of returns on investment (ROI), given that ROI equals profit divided by budget.

Furthermore, I set $\delta = 1$ for each publisher j. The reason is that advertisement auctions are held frequently in practice, which results in discount factors that approximate 1.

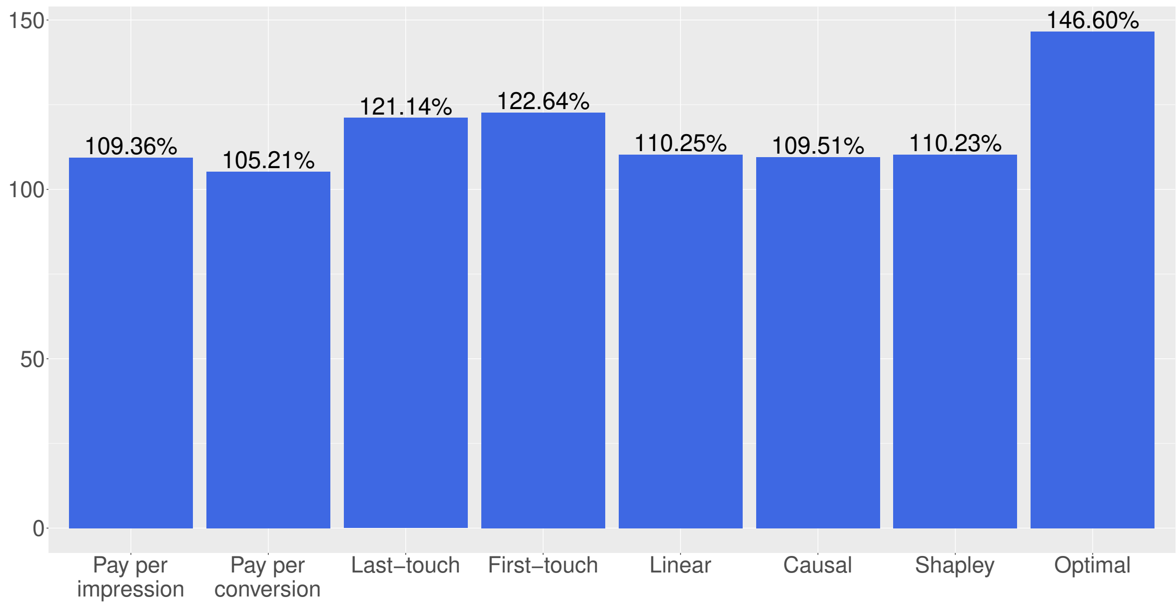

Figure 2: Return on investment (ROI) under different incentive schemes

Note: This figure compares the ROI under different incentive schemes, expressed as percentages. The incentive schemes include pay-per-impression (advertisers pay for each ad impression), pay-per-conversion (advertisers pay only when a user converts), last-touch (attributes conversion to the last publisher in the touchpoint sequence), first-touch (attributes conversion to the first publisher in the touchpoint sequence), linear (attributes conversion equally to all publishers in the touchpoint sequence), causal (attributes conversion based on estimated causal effects), Shapley (attributes conversion using Shapley values), and the optimal incentive scheme. The height of each bar represents the ROI.

Figure 2 plots the counterfactual ROI under different incentive schemes. One can group the ROIs into three distinct levels. The lowest ROI level consists of the pay-per-impression scheme, the pay-per-conversion scheme, the linear attribution algorithm, the causal attribution algorithm, and the Shapley attribution algorithm. These methods share a common feature in that they are multi-touch attribution algorithms, where all publishers involved in a touchpoint sequence are credited or compensated. This form of incentives can be contrasted with the optimal incentive scheme on the highest ROI level, where a publisher is compensated only when the touchpoint sequence involves solely itself.

The last-touch and first-touch attribution algorithms, which exist in the medium ROI tier, bridge the two types of strategies. These two algorithms are single-touch attribution algorithms such that when a touchpoint sequence involves multiple publishers, exactly one publisher gets the credit, contrasting with the multi-touch approach where all involved publishers get credit or the optimal scheme where none does.

In summary, the counterfactual analysis reveals the profit advantages of the optimal incentive scheme. Also, the single-touch approaches outperform the multi-touch ones, the intuition being that the former creates incentives that align better with the optimal incentives.

6 Summary

This paper studies the strategic implications of online advertisers adopting attribution algorithms. I develop a dynamic model capturing the interactions between advertisers and publishers and show how attribution algorithms create dynamic incentives for publishers. These incentives resemble pay-per-action mechanisms with contingent actions based on the advertisement impression sequence and conversion outcome. Moreover, I characterize the advertiser's optimal strategy. Counterfactual simulations demonstrate that the optimal strategy increases the advertiser's ROI on the order of 20–40% compared to standard attribution algorithms. Furthermore, the findings indicate that single-touch attribution algorithms could yield higher profits than multi-touch attribution algorithms.

The results of this study have important implications for advertisers. It highlights the crucial role of dynamic incentives in attribution design. Failing to account for these incentives could potentially lead to suboptimal bidding strategies and lower ROI. By carefully designing incentives, advertisers can align publisher behavior with their objectives, leading to more effective advertising strategies.

While this study provides valuable insights, it is important to acknowledge its limitations. As discussed in Sections 2 and 3, the model makes several simplifying assumptions regarding user browsing behavior, which could be relaxed in future research.

In conclusion, this paper contributes to the growing literature on the economics of AI by examining the strategic implications of attribution algorithms. The findings underscore the importance of considering dynamic incentives in attribution design and provide actionable insights for advertisers seeking to optimize their advertising performance. As the online advertising industry continues to evolve, a deeper understanding of the interplay between measurement and incentives will be crucial for developing effective strategies and fostering a healthy ecosystem for all stakeholders.

References

Abhishek, V., Despotakis, S., and Ravi, R. (2017). Multi-channel attribution: The blind spot of online advertising. *Available at SSRN 2959778*. 6

Barajas, J., Akella, R., Holtan, M., and Flores, A. (2016). Experimental designs and estimation for online display advertising attribution in marketplaces. *Marketing Science*, 35(3):465–483. 6

Berman, R. (2018). Beyond the last touch: Attribution in online advertising. *Marketing Science*, 37(5):771–792. 6

Blake, T., Nosko, C., and Tadelis, S. (2015). Consumer heterogeneity and paid search effectiveness: A large-scale field experiment. *Econometrica*, 83(1):155–174. 6

California Privacy Rights Act (2020). Cal. Civ. Code § 1798.100–1798.199. 2

Danaher, P. J. and van Heerde, H. J. (2018). Delusion in attribution: Caveats in using attribution for multimedia budget allocation. *Journal of Marketing Research*, 55(5):667–685. 6

Du, R., Zhong, Y., Nair, H., Cui, B., and Shou, R. (2019). Causally driven incremental multi touch attribution using a recurrent neural network. *arXiv preprint arXiv:1902.00215*. 6

D'Annunzio, A. and Russo, A. (2023). Intermediaries in the online advertising market. *Marketing Science*. 2

European Parliament and Council of the European Union (2018). Regulation (EU) 2016/679 of the European Parliament and of the Council of 27 April 2016 on the protection of natural persons with regard to the processing of personal data and on the free movement of such data, and repealing Directive 95/46/EC (General Data Protection Regulation). Official Journal of the European Union, L119. 2

Fudenberg, D. and Villas-Boas, J. M. (2006). Behavior-based price discrimination and customer recognition. *Handbook on Economics and Information Systems*, 1:377–436. 6

Gordon, B. R., Zettelmeyer, F., Bhargava, N., and Chapsky, D. (2019). A comparison of approaches to advertising measurement: Evidence from big field experiments at facebook. *Marketing Science*, 38(2):193–225. 6

Green, J. R. and Stokey, N. L. (1983). A comparison of tournaments and contracts. *Journal of Political Economy*, 91(3):349–364. 6, 22

Heckman, J. J. (1979). Sample selection bias as a specification error. *Econometrica*, pages 153–161. 20, 29

IAB Technology Lab (2016). OpenRTB API specification version 2.5. 2

Johnson, G. A., Shriver, S. K., and Goldberg, S. G. (2023). Privacy and market concentration: intended and unintended consequences of the gdpr. *Management Science*. 2

Ke, T. T. and Sudhir, K. (2022). Privacy rights and data security: Gdpr and personal data markets. *Management Science*. 2

Kuksov, D. and Villas-Boas, J. M. (2019). The performance measurement trap. *Marketing Science*, 38(1):68–87. 6

Lazear, E. P. and Rosen, S. (1981). Rank-order tournaments as optimum labor contracts. *Journal of Political Economy*, 89(5):841–864. 6, 22

Lewis, R. A., Rao, J. M., and Reiley, D. H. (2011). Here, there, and everywhere: correlated online behaviors can lead to overestimates of the effects of advertising. In *Proceedings of the 20th International Conference on World Wide Web*, pages 157–166. 6

Li, H. and Kannan, P. (2014). Attributing conversions in a multichannel online marketing environment: An empirical model and a field experiment. *Journal of Marketing Research*, 51(1):40–56. 6

Li, H., Kannan, P., Viswanathan, S., and Pani, A. (2016). Attribution strategies and return on keyword investment in paid search advertising. *Marketing Science*, 35(6):831–848. 6

Marotta, V., Wu, Y., Zhang, K., and Acquisti, A. (2022). The welfare impact of targeted advertising technologies. *Information Systems Research*, 33(1):131–151. 2

Milgrom, P. R. and Weber, R. J. (1982). A theory of auctions and competitive bidding. *Econometrica*, pages 1089–1122. 5, 23

Nalebuff, B. J. and Stiglitz, J. E. (1983). Prizes and incentives: towards a general theory of compensation and competition. *The Bell Journal of Economics*, pages 21–43. 6, 22

Shapley, L. S. (1953). A value for n-person games. 12

Weintraub, G. Y., Benkard, C. L., and Van Roy, B. (2008). Markov perfect industry dynamics with many firms. *Econometrica*, 76(6):1375–1411. 14

Xu, L., Duan, J. A., and Whinston, A. (2014). Path to purchase: A mutually exciting point process model for online advertising and conversion. *Management Science*, 60(6):1392–1412. 6

Zhang, J. (2011). The perils of behavior-based personalization. *Marketing Science*, 30(1):170–186. 6

A Institutional Background

In this appendix, I provide additional information regarding the institutional background of online display advertising.

Real-time bidding (RTB) is one of the most prevalent ways for advertisers to buy advertisement impressions. Unlike traditional advertising, where advertisement slots are bought in bulk, RTB enables advertisers to bid for individual impressions in real-time. Auctions often wrap up in just milliseconds.

A salient feature in RTB is "header bidding." When a user visits a publisher's website, header bidding solicits bids from various demand sources and allows multiple demand sources to bid simultaneously. This simultaneous bidding typically occurs in the code located within a web page's header, which is how the term was coined.[33]

In the next subsections, I provide more information regarding the advertisers' bidding rules, the publishers' control over advertisement display, and the pay-per-action mechanisms with contingent conversion actions.

A.1 A Simple Bidding Model

Advertisers typically participate in first-price, per-impression auctions, where the auction winner pays their bid amount if the publisher displays the advertisement. To determine the optimal bid, advertisers often rely on heuristic models that estimate the expected profits attributed to each display.

Let b be the advertiser's per-impression bid and $W(b)$ be the win rate as a function of the bid. The advertiser's estimated incremental conversion rate is denoted as r, and the value of a conversion is denoted as K. The expected profits attributed to a single ad display can be expressed as

$$(K \cdot r - b) \cdot W(b).$$

This heuristic model can be seen as an approximation of the optimization problem (10) in Section 3.2, which involves incentive schemes for all J publishers. The heuristic model decomposes the problem into J separate heuristic models, one for each publisher, with the advertiser maximizing the expected attributed profits with respect to the bid b for each publisher.

Under regularity conditions, the optimal bid b^* is an increasing function of the estimated conversion rate r, that is, $b^* = \psi(r)$. When $W(b)$ has constant elasticity, $\psi(r)$ is a linear function. In practice, this linear bidding rule is known as target CPA bidding,

[33] As technological advancements continue to shape the industry, the methods of header bidding have evolved. Traditional methods, known as client-side header bidding, directly involve the user's browser. However, there is a growing trend towards server-side header bidding. In this model, bidding is delegated to a server, which can more efficiently manage multiple bid requests and responses. This server-side approach is gaining popularity because of its ability to minimize latency for the user and the server's enhanced capacity to run sophisticated yield optimization algorithms.

where the bid is a linear function of the advertising effectiveness measure. This simple bidding model provides a micro-foundation for the bidding rules observed in the industry.

A.2 Publisher Advertisement Display Decisions

The rise of header bidding has significantly impacted the role of publishers in the advertising ecosystem. Publishers have significant discretion in determining which advertisements to display to users. During header bidding, publishers receive bids from multiple demand sources but do not necessarily select the highest bidder. Instead, they employ sophisticated yield optimization algorithms to maximize their long-term revenue.

These algorithms consider a wide range of factors when making advertisement display decisions, including (i) advertiser characteristics: historical performance, reputation, and relationship with the publisher; (ii) advertisement attributes: content, format, and user engagement potential; (iii) contextual factors: page content, time of day, and other environmental factors; and (iv) user-specific data: browsing history, demographics, and personal attributes.

To account for these factors, publishers apply "modifiers" to the received bids. These modifiers, which can be additive or multiplicative, adjust the raw bid values based on the factors. The bids are then ranked according to the adjusted values rather than the raw bid amounts.

This procedure allows publishers to exploit their informational advantage and target advertisements to specific users in ways that maximize their own benefits. Such strategic behavior may not always be the result of deliberate manipulation but can naturally arise from the use of yield optimization algorithms designed to maximize long-term revenue.

The flexibility that publishers have in determining which advertisements to display, combined with their access to granular user data, could create potential moral hazard issues.

A.3 Pay-per-action with Contingent Conversion Actions

Pay-per-action (PPA) mechanisms provide advertisers with the flexibility to define desired user actions, such as site visits or purchases, as conversion actions and place bids accordingly. Under a PPA mechanism, the advertiser only pays the publisher when a conversion action is completed.

Importantly, the PPA mechanism allows for the design of sophisticated bidding strategies without requiring market power or monopoly power. Advertisers can define conversion actions based on a user's touchpoint sequence. For example, upon a user's purchase, the advertiser can verify whether the user's preceding sequence of touchpoints involves only a single publisher or multiple publishers. This mechanism enables the advertiser to compensate publishers differently in each scenario, creating an incentive scheme.

To implement PPA mechanisms with contingent actions, the advertiser defines conversion actions over all possible combinations of touchpoints ω and conversion outcomes y. Denote $g_j(\omega, y)$ as the payment to publisher j when the touchpoint is ω and the conversion outcome is y, then the advertiser can implement this compensation scheme by bidding $g_j(\omega, y)$ on publisher j for all possible (ω, y) tuples.

This flexibility in defining conversion actions and adjusting bids based on touchpoint sequences allows advertisers to create incentive schemes that align with their objectives and mitigate potential moral hazard issues arising from publishers' strategic behavior.

B Extensions

B.1 Multiple Advertisers

In this appendix, I consider an extension that relaxes the assumption that there is a single advertiser and other advertisers' actions are held constant. I then show similar strategic effects of the publishers as in Section 2.

The game consists of K advertisers and J publishers. As in the main text, everything is conditional on all publicly observed user characteristics, in other words, the model considered here is conditional on a specific user segment. Every advertiser can bid on every publisher.[34] In the model setup, I will emphasize more on the difference between the model presented here and in the main text.

In the stage game, advertiser k submits a per-impression bid $b_j^{(k)}$ to publisher j. Then users arrive, where user i has a browsing history H_i, and there is a cost $\xi_{ij}^{(k)}$ if publisher j displays advertiser k's advertisement to user i. This cost captures the potential negative impact on the publisher's reputation caused by a low-quality advertisement. The value of displaying no advertisements is normalized to zero.

In the stage game, after the publishers make the display decisions, the touchpoint sequence $T_i^{(k)}$ and conversion outcome $Y_i^{(k)}$ are realized. If publisher j displays advertiser k's advertisement to user i, publisher j will appear in the sequence $T_i^{(k)}$ and receive a payment $b_j^{(k)}$ at the end of the period. I assume that for a given user, whether this user converts on advertiser k does not depend on the touchpoint sequence of other advertisers. This assumption is closer to the settings where the advertisers are from different industries, instead of competing in the same industry. It could be interesting to consider scenarios where displaying one advertiser's advertisement will negatively affect other advertisers' conversion outcomes, but this is beyond the scope of this paper.

The dynamic game G is an infinite repetition of the stage game among the publishers, with δ as the common discount factor. Denote the advertisers' bid vector in period t as

[34] In the real world, some advertisers do not advertise on some publishers, which is captured in this model by the advertiser submitting sufficiently low bids to these publishers.

$\mathbf{b}^{(t)}$, with $b_j^{(kt)}$ being advertiser k's bid for publisher j. At the beginning of the game, the K advertisers simultaneously set their respective bids and attribution algorithms. In period t, the stage game is played. Each publisher makes the advertisement display decisions, then the touchpoint sequence and conversions are realized and the payments are received accordingly. Then, advertiser k observes $\pi^{(kt)}$, the distribution of user touchpoint sequences and conversion outcomes. Denote $\Psi^{(k)}$ as advertiser k's algorithm, which takes $b^{(t)}$ and $\pi^{(t)}$ as inputs, that is,

$$b^{(k,t+1)} = \Psi^{(k)}(b^{(kt)}, \pi^{(kt)}).$$

The game is denoted as $G(\Psi)$, where $\Psi = (\Psi^{(k)})$ is the set of the advertisers' bidding rules.

Now drop the time superscript to simplify the notation. In each period, publisher j's decision rule can be denoted as $a_j = (a_{ij}^{(k)})$, where $a_{ij}^{(k)}$ is the indicator of whether publisher i display advertiser k's advertisement to user i, then it satisfies the constraint that for all i, j,

$$\sum_{k=1}^{K} a_{ij}^{(k)} \leq 1.$$

The oblivious equilibrium of the game can be defined similarly as in Section 2.4. Analogously, the state variable for publisher j is the vector of bids submitted by all advertisers, $\mathbf{b}_j = (b_j^{(k)})$, and the corresponding Bellman equation is

$$\begin{aligned} V_j(\mathbf{b}_j) = \max_{a_j} \quad & \sum_{k=1}^{K} \int_0^1 a_{ij}^{(k)}(b_j^{(k)} - \xi_{ij}^{(k)})\,\mathrm{d}i + \delta \cdot V_j(\mathbf{b}_j'), \\ \text{s.t.} \quad & \textstyle\sum_{k=1}^{K} a_{ij}^{(k)} \leq 1, \forall i, j. \end{aligned}$$

Moreover, the next period's bid vector $\mathbf{b}_j'$ is determined by the algorithms $\Psi^{(k)}$'s. Denote the state-action value function

$$Q_j(\mathbf{b}_j, a_j) \equiv \sum_{k=1}^{K} \int a_{ij}^{(k)}(b_j^{(k)} - \xi_{ij}^{(k)})\,\mathrm{d}i + \delta \cdot V_j(\mathbf{b}_j'),$$

then the best-response display strategy a_j^* satisfies

$$a_{ij}^{(k)*} = \mathbb{1}\left(\frac{\partial Q_j}{\partial a_{ij}^{(k)}} \geq 0, \frac{\partial Q_j}{\partial a_{ij}^{(k)}} \geq \frac{\partial Q_j}{\partial a_{ij}^{(k')}}, \forall k' \neq k\right),$$

that is, when deciding which advertisement to show, publisher j will display advertiser k's advertisement if displaying it is profitable in term of long-term value ($\frac{\partial Q_j}{\partial a_{ij}^{(k)}} \geq 0$) and the impact on the long-term value is highest among all advertisers.

The extended static game can also defined similarly as in Section 2.4. Specifically, let $\tilde{g} = (\tilde{g}^{(k)})$ be the set of incentive schemes utilized by each of the advertisers. A similar static equivalence result can then be shown below.

Proposition B.1. *Suppose Ψ is a set of bidding rules such that for each advertiser k and publisher j, $\Psi_j^{(k)}$ is homogeneous of degree 0 in $\pi_j^{(k)}$. Let $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ denote an oblivious equilibrium of the dynamic game $G(\Psi)$, and $\boldsymbol{a}^*|_{\boldsymbol{b}^*}$ denote the publishers' strategy profile when the vector of bids is $\boldsymbol{b}^*$. Then there exists an set of incentive scheme $\tilde{g}$ such that:*

1. *The strategy profile $\boldsymbol{a}^*|_{\boldsymbol{b}^*}$ is a Bayesian Nash equilibrium of the static game $\Gamma(\tilde{g})$.*
2. *The equilibrium payoffs of the advertiser and the publishers under $\boldsymbol{a}^*|_{\boldsymbol{b}^*}$ in the static game $\Gamma(\tilde{g})$ equal their equilibrium payoffs under $(\boldsymbol{a}^*, \boldsymbol{b}^*)$ in the dynamic game $G(\Psi)$.*

This proposition means that, the dynamic incentives can also be replicated by static contract-like incentives in this model extension.

One may be interested in whether and how the results in Section 3 generalize to this extension. First, note that the model considered here can be seen as micro-founding the opportunity cost in the main model. The model in the main text assumes that for a focal advertiser k, there is an opportunity cost $c_{ij}^{(k)}$ of displaying the advertisement. It can be shown that

$$c_{ij}^{(k)} = \xi_{ij}^{(k)} + \max\left\{0, \max_{k' \neq k}\{b_j^{(k')*} - \xi_{ij}^{(k')}\}\right\},$$

where $b_j^{(k')*}$ is the stationary bid that advertiser k submits to publisher j in the equilibrium.

Then, the analysis in Section 3 can be seen as a best-response analysis of a focal advertiser, holding constant other advertisers' actions. If some conditions regarding the primitives ensure that for each advertiser k, the opportunity costs $(c_{ij}^{(k)})$ are always affiliated across the publishers, one could use Proposition 3.2 to show that all advertisers adopt per-conversion and competitive incentive schemes in the equilibrium.[35] Investigating such conditions regarding the primitives is beyond the scope of the paper but could be an interesting future direction.

C Proofs

C.1 Proof of Proposition 2.1

The key to the proof is to evaluate $\dfrac{\partial Q_j}{\partial a_{ij}}$. To simplify notation, every quantity below is conditional on publisher j's observed information, the user browsing history H and

[35] Note that appropriate assumptions on the conversion rate function are needed as well.

opportunity cost c_j. By the chain rule of differentiation,

$$\frac{\partial Q_j}{\partial a_{ij}} = (b_j^* - c_{ij}) + \delta \cdot \frac{\partial V_j(\psi_j(b_j^*, r_j))}{\partial a_{ij}} = (b_j^* - c_{ij}) + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial a_{ij}}.$$

Given that r_j depends on π_j, a vector consisting of $\pi_{\omega,y}$ with $j \in \omega$, we have

$$\frac{\partial \psi_j(b_j^*, r_j)}{\partial a_{ij}} = \left(\frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_j}\right)^{\mathsf{T}} \left(\frac{\partial \pi_j}{\partial a_{ij}}\right) = \sum_{(\omega,y):j\in\omega} \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \cdot \frac{\partial \pi_{\omega,y}}{\partial a_{ij}}.$$

For $j \in \omega$, we have

$$\pi_{\omega,y} = \mathbb{P}(T = \omega, Y = y) = \int a_{ij} \cdot \mathbb{1}(T_i(1, \mathbf{a}_{i,-j}^*) = \omega, Y_i(1, \mathbf{a}_{i,-j}^*) = y)\, \mathrm{d}i,$$

which leads to

$$\frac{\partial \pi_{\omega,y}}{\partial a_{ij}} = \mathbb{1}(T_i(1, \mathbf{a}_{i,-j}^*) = \omega, Y_i(1, \mathbf{a}_{i,-j}^*) = y).$$

Additionally, since

$$\sum_{(\omega,y):j\in\omega} \mathbb{1}(T_i(1, \mathbf{a}_{i,-j}^*) = \omega, Y_i(1, \mathbf{a}_{i,-j}^*) = y) = 1,$$

we can obtain

$$\begin{aligned}
&\frac{\partial Q_j}{\partial a_{ij}} \\
&= (b_j^* - c_{ij}) + \delta \cdot V_j'(b_j^*) \cdot \sum_{(\omega,y):j\in\omega} \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \cdot \mathbb{1}(T_i(1, \mathbf{a}_{i,-j}^*) = \omega, Y_i(1, \mathbf{a}_{i,-j}^*) = y) \\
&= -\, c_{ij} + \sum_{(\omega,y):j\in\omega} \left(b_j^* + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}}\right) \cdot \mathbb{1}(T_i(1, \mathbf{a}_{i,-j}^*) = \omega, Y_i(1, \mathbf{a}_{i,-j}^*) = y).
\end{aligned}$$

This means that the incentive in this dynamic scheme equals a static scheme $\tilde{g}$ defined as

$$\tilde{g}_j(\omega, y) = b_j^* + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}}. \tag{13}$$

Denote publisher j's equilibrium display strategy under the stationary bid b_j^* in the dynamic model as $a_j^*|_{b_j^*}$, then since the incentives are the same in a statics game under scheme $\tilde{g}$, $a_j^*|_{b_j^*}$ is also the best response to $a_{-j}^*|_{b_{-j}}$ in the static game. Thus, the first part of the proposition holds.

To show the second part of the proposition, it suffices to show that for each publisher, the advertising expenditures are the same under both the static scheme g and the dynamic scheme $\tilde{g}$. That is, it is sufficient to show that for each publisher j,

$$\mathbb{E}[g_j(T(\mathbf{a}^*|_{b^*}), Y(\mathbf{a}^*|_{b^*}))] = \mathbb{E}[\tilde{g}_j(T(\mathbf{a}^*|_{b^*}), Y(\mathbf{a}^*|_{b^*}))].$$

Since

$$\mathbb{E}[g_j(T(\mathbf{a}^*|_{b^*}), Y(\mathbf{a}^*|_{b^*}))] = \int a_{ij} \cdot b_j^* \,\mathrm{d}i,$$

and

$$\begin{aligned}
&\mathbb{E}[\tilde{g}_j(T(\mathbf{a}^*|_{b^*}), Y(\mathbf{a}^*|_{b^*}))] \\
=&\int a_{ij} \cdot \sum_{(\omega,y):j\in\omega} \left(b_j^* + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \right) \cdot \mathbb{1}(T_i(1, \mathbf{a}^*_{i,-j}) = \omega, Y_i(1, \mathbf{a}^*_{i,-j}) = y) \,\mathrm{d}i,
\end{aligned}$$

the difference amounts to

$$\begin{aligned}
&\mathbb{E}[\tilde{g}_j(T(\mathbf{a}^*|_b), Y(\mathbf{a}^*|_b))] - \mathbb{E}[g_j(T(\mathbf{a}^*|_b), Y(\mathbf{a}^*|_b))] \\
=&\,\delta \cdot V_j'(b_j^*) \cdot \sum_{(\omega,y):j\in\omega} \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \cdot \int a_{ij} \cdot \mathbb{1}(T_i(1, \mathbf{a}^*_{i,-j}) = \omega, Y_i(1, \mathbf{a}^*_{i,-j}) = y) \,\mathrm{d}i \\
=&\,\delta \cdot V_j'(b_j^*) \cdot \sum_{(\omega,y):j\in\omega} \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \cdot \pi_{\omega,y}.
\end{aligned}$$

Recall that we assume r_j is homogeneous of degree 0 in $\pi_{\omega,y}$'s, which implies that $\psi_j(b_j^*, r_j)$ is also homogeneous of degree 0 in $\pi_{\omega,y}$'s. By Euler's homogeneous function theorem, it follows that

$$\sum_{(\omega,y):j\in\omega} \frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \cdot \pi_{\omega,y} = 0,$$

which then gives us

$$\mathbb{E}[g_j(T(\mathbf{a}^*|_{b^*}), Y(\mathbf{a}^*|_{b^*}))] = \mathbb{E}[\tilde{g}_j(T(\mathbf{a}^*|_{b^*}), Y(\mathbf{a}^*|_{b^*}))].$$

Finally, we need to show that $\tilde{g}_j(\omega, y) \geq 0$ and $\tilde{g}_j(\omega, 1) \geq \tilde{g}_j(\omega, 0)$ for any ω such that $j \in \omega$. By stationarity, $\psi_j(b_j^*, r_j) = b_j^*$, then applying the envelop theorem to (5) yields

$$V_j'(b_j^*) = \int a_{ij}^* \,\mathrm{d}i + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j},$$

so

$$V_j'(b_j^*) = \frac{\int a_{ij}^* \,\mathrm{d}i}{1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}} = \frac{\sum_{\omega:j\in\omega} \pi_\omega}{1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}}. \tag{14}$$

Also, recalling the assumption that $\frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j} \leq 1$ and $-\frac{\partial \psi_j(b_j^*, r_j)}{\partial \pi_{\omega,y}} \cdot \frac{\sum_{\omega:j\in\omega} \pi_\omega}{\psi_j(b_j^*, r_j)} \leq$

$1 - \dfrac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}$, we can deduce

$$\begin{aligned}
&\tilde{g}_j(\omega, y) \\
\geq &\, b_j^* - \delta \cdot \frac{\sum_{\omega: j \in \omega} \pi_\omega}{1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}} \cdot \frac{b_j^*}{\sum_{\omega: j \in \omega} \pi_\omega} \left(1 - \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}\right) \\
= &\left(1 - \delta \cdot \frac{1 - \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}}{1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial b_j}}\right) b_j^* \\
\geq &\, 0.
\end{aligned}$$

Also,

$$\tilde{g}_j(\omega, 1) - \tilde{g}_j(\omega, 0) = \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j)}{\partial r_j} \cdot \left(\frac{\partial r_j}{\partial \pi_{\omega,1}} - \frac{\partial r_j}{\partial \pi_{\omega,0}}\right).$$

Since $V_j'(b_j^*) \geq 0$, and by the assumptions, $\dfrac{\partial \psi_j(b_j^*, r_j)}{\partial r_j} \geq 0$ and $\dfrac{\partial r_j}{\partial \pi_{\omega,1}} \geq \dfrac{\partial r_j}{\partial \pi_{\omega,0}}$, so we have $\tilde{g}_j(\omega, 1) - \tilde{g}_j(\omega, 0) \geq 0$.

C.2 Proof of Proposition 2.2

According to (13) and (14), it suffices to evaluate $\dfrac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial \pi_{\omega,y}}$. By the chain rule,

$$\frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial \pi_{\omega,y}} = \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial r_j} \cdot \frac{\partial r_j^{\mathrm{RB}}}{\partial \pi_{\omega,y}},$$

and according to (2),

$$\frac{\partial r_j^{\mathrm{RB}}}{\partial \pi_{\omega,y}} = \begin{cases} \frac{\chi(j,\omega) - r_j^{\mathrm{RB}}}{\sum_{\omega: j \in \omega} \pi_\omega}, & y = 1 \\ \frac{-r_j^{\mathrm{RB}}}{\sum_{\omega: j \in \omega} \pi_\omega}, & y = 0 \end{cases}.$$

Plugging this equation and (14) into (13) yields

$$\begin{aligned}
&\tilde{g}_j^{\mathrm{RB}}(\omega, y) \\
= &\, b_j^* + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial \pi_{\omega,y}} \\
= &\, b_j^* + \delta \cdot \frac{\sum_{\omega: j \in \omega} \pi_\omega}{1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial b_j}} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial r_j} \cdot \frac{\chi(j,\omega) \cdot \mathbb{1}(y=1) - r_j^{\mathrm{RB}}}{\sum_{\omega: j \in \omega} \pi_\omega} \\
= &\, b_j^* + \delta \cdot \left(1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial r_j} \cdot \left(\chi(j,\omega) \cdot \mathbb{1}(y=1) - r_j^{\mathrm{RB}}\right)
\end{aligned}$$

$$
\begin{aligned}
&= b_j^* + \underbrace{\delta \cdot \left(1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{RB}})}{\partial r_j} \cdot \frac{r_j^{\mathrm{RB}}}{b_j^*}}_{=\eta_j} \cdot \left(\underbrace{\frac{b_j^*}{r_j^{\mathrm{RB}}}}_{=v_j^*} \cdot \chi(j,\omega) \cdot \mathbb{1}(y=1) - b_j^* \right) \\
&= b_j^* + \eta_j \cdot (v_j^* \cdot \chi(j,\omega) \cdot \mathbb{1}(y=1) - b_j^*) \\
&= (1-\eta_j) \cdot b_j^* + \eta_j \cdot v_j^* \cdot \chi(j,\omega) \cdot \mathbb{1}(y=1).
\end{aligned}
$$

C.3 Proof of Proposition 2.3

Analogous to the proof of Proposition 2.2, it suffices to evaluate $\frac{\partial r_j^{\mathrm{C}}}{\partial \pi_{\omega,y}}$. According to (3),

$$
\frac{\partial r_j^{\mathrm{C}}}{\partial \pi_{\omega,y}} = \begin{cases} \frac{1-\rho_{\omega\setminus\{j\}} - r_j^{\mathrm{C}}}{\sum_{\omega:j\in\omega} \pi_\omega}, & y=1 \\ \frac{-\rho_{\omega\setminus\{j\}} - r_j^{\mathrm{C}}}{\sum_{\omega:j\in\omega} \pi_\omega}, & y=0 \end{cases}.
$$

Then we have

$$
\begin{aligned}
&\tilde{g}_j^{\mathrm{C}}(\omega,y) \\
&= b_j^* + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial \pi_{\omega,y}} \\
&= b_j^* + \delta \cdot \frac{\sum_{\omega:j\in\omega} \pi_\omega}{1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial b_j}} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial r_j} \cdot \frac{\mathbb{1}(y=1) - \rho_{\omega\setminus\{j\}} - r_j^{\mathrm{C}}}{\sum_{\omega:j\in\omega} \pi_\omega} \\
&= b_j^* + \delta \cdot \left(1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial r_j} \cdot \left(\mathbb{1}(y=1) - \rho_{\omega\setminus\{j\}} - r_j^{\mathrm{C}}\right) \\
&= b_j^* + \underbrace{\delta \cdot \left(1 - \delta \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{C}})}{\partial r_j} \cdot \frac{r_j^{\mathrm{C}}}{b_j^*}}_{=\eta_j} \cdot \left(\underbrace{\frac{b_j^*}{r_j^{\mathrm{C}}}}_{=v_j^*} \cdot (\mathbb{1}(y=1) - \rho_{\omega\setminus\{j\}}) - b_j^* \right) \\
&= b_j^* + \eta_j \cdot (v_j^* \cdot (\mathbb{1}(y=1) - \rho_{\omega\setminus\{j\}}) - b_j^*) \\
&= (1-\eta_j) \cdot b_j^* + \eta_j \cdot v_j^* \cdot (\mathbb{1}(y=1) - \rho_{\omega\setminus\{j\}}).
\end{aligned}
$$

C.4 Proof of Proposition 2.4

Analogous to the proof of Proposition 2.2, it suffices to evaluate $\frac{\partial r_j^{\mathrm{S}}}{\partial \pi_{\omega,y}}$. According to (4),

$$
\frac{\partial r_j^{\mathrm{S}}}{\partial \pi_{\omega,y}} = \frac{\sum_{\tilde{\omega}:j\in\tilde{\omega}} \frac{\partial \varphi(j,\tilde{\omega})}{\partial \pi_{\omega,y}} \cdot \pi_{\tilde{\omega}} + \varphi(j,\omega) - r_j^{\mathrm{S}}}{\sum_{\omega:j\in\omega} \pi_\omega}.
$$

Since

$$\varphi(j,\tilde{\omega}) = \sum_{\omega':j\in\omega'\subseteq\tilde{\omega}} \frac{(|\omega'|-1)!\,(|\tilde{\omega}|-|\omega'|)!}{|\tilde{\omega}|!} \cdot (\rho_{\omega'} - \rho_{\omega'\setminus\{j\}}),$$

for $\tilde{\omega}$ such that $\omega \subseteq \tilde{\omega}$ holds,

$$\frac{\partial \varphi(j,\tilde{\omega})}{\partial \pi_{\omega,y}} = \frac{(|\omega|-1)!\,(|\tilde{\omega}|-|\omega|)!}{|\tilde{\omega}|!} \cdot \frac{\partial \rho_\omega}{\partial \pi_{\omega,y}} = \begin{cases} \frac{(|\omega|-1)!(|\tilde{\omega}|-|\omega|)!}{|\tilde{\omega}|!} \cdot \frac{1-\rho_\omega}{\pi_\omega}, & y=1 \\ \frac{(|\omega|-1)!(|\tilde{\omega}|-|\omega|)!}{|\tilde{\omega}|!} \cdot \frac{-\rho_\omega}{\pi_\omega}, & y=0 \end{cases},$$

while for $\tilde{\omega}$ such that $\omega \subseteq \tilde{\omega}$ does not hold, $\dfrac{\partial \varphi(j,\tilde{\omega})}{\partial \pi_{\omega,y}} = 0$. Thus, we have

$$\frac{\partial r_j^{\mathrm{S}}}{\partial \pi_{\omega,y}} = \begin{cases} \frac{\zeta_j(\omega)\cdot(1-\rho_\omega)+\varphi(j,\omega)-r_j^{\mathrm{S}}}{\sum_{\omega:j\in\omega}\pi_\omega}, & y=1 \\ \frac{-\zeta_j(\omega)\cdot\rho_\omega+\varphi(j,\omega)-r_j^{\mathrm{S}}}{\sum_{\omega:j\in\omega}\pi_\omega}, & y=0 \end{cases},$$

where

$$\zeta_j(\omega) = \sum_{\tilde{\omega}:\omega\subseteq\tilde{\omega}} \frac{(|\omega|-1)!\,(|\tilde{\omega}|-|\omega|)!}{|\tilde{\omega}|!} \cdot \frac{\pi_{\tilde{\omega}}}{\pi_\omega}.$$

Then we have

$$\begin{aligned}
&\tilde{g}_j^{\mathrm{S}}(\omega,y) \\
&= b_j^* + \delta \cdot V_j'(b_j^*) \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{S}})}{\partial \pi_{\omega,y}} \\
&= b_j^* + \delta \cdot \frac{\sum_{\omega:j\in\omega}\pi_\omega}{1-\delta\cdot\frac{\partial\psi_j(b_j^*,r_j^{\mathrm{S}})}{\partial b_j}} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{S}})}{\partial r_j} \cdot \frac{(\zeta_j(\omega)\cdot(\mathbb{1}(y=1)-\rho_\omega)+\varphi(j,\omega))-r_j^{\mathrm{S}}}{\sum_{\omega:j\in\omega}\pi_\omega} \\
&= b_j^* + \delta \cdot \left(1-\delta\cdot\frac{\partial\psi_j(b_j^*,r_j^{\mathrm{S}})}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{S}})}{\partial r_j} \cdot \left((\zeta_j(\omega)\cdot(\mathbb{1}(y=1)-\rho_\omega)+\varphi(j,\omega))-r_j^{\mathrm{S}}\right) \\
&= b_j^* + \underbrace{\delta \cdot \left(1-\delta\cdot\frac{\partial\psi_j(b_j^*,r_j^{\mathrm{S}})}{\partial b_j}\right)^{-1} \cdot \frac{\partial \psi_j(b_j^*, r_j^{\mathrm{S}})}{\partial r_j} \cdot \frac{r_j^{\mathrm{S}}}{b_j^*}}_{=\eta_j} \cdot \left(\underbrace{\frac{b_j^*}{r_j^{\mathrm{S}}}}_{=v_j^*} \cdot (\zeta_j(\omega)\cdot(\mathbb{1}(y=1)-\rho_\omega)+\varphi(j,\omega)) - b_j^* \right) \\
&= b_j^* + \eta_j \cdot (v_j^* \cdot (\zeta_j(\omega)\cdot(\mathbb{1}(y=1)-\rho_\omega)+\varphi(j,\omega)) - b_j^*) \\
&= (1-\eta_j)\cdot b_j^* + \eta_j \cdot v_j^* \cdot (\zeta_j(\omega)\cdot(\mathbb{1}(y=1)-\rho_\omega)+\varphi(j,\omega)).
\end{aligned}$$

C.5 Proof of Propositions 3.1 and 3.2

I first prove Proposition 3.1. Proposition 3.2 is essentially an extension of Proposition 3.1 in scenarios where the conversion rate is a function of the opportunity costs, so I elucidate how the techniques are adapted to prove Proposition 3.2.

C.5.1 Proof of Proposition 3.1

Suppose the opportunity cost distribution f exhibit affiliation. The goal is to show that the optimal scheme g is competitive. The proof proceeds by contradiction. Specifically, suppose that for publisher j, touchpoint sequence ω such that $\omega_0 \equiv (j) \subset \omega$, and conversion indicator y, $g_j(\omega, y) > 0$ holds, then it is possible to construct a scheme $\bar{g}$ that yields a higher payoff for the advertiser.

Before constructing $\bar{g}$, here I define some notations. For publisher k, denote $\bar{c}_k$ as the marginal user type in equilibrium under scheme g, which satisfies

$$R_k(\bar{c}_k; \bar{\mathbf{c}}_{-k}) = \bar{c}_k. \tag{15}$$

In this equation, $R_k(\bar{c}_k; \bar{\mathbf{c}}_{-k})$ is the expected revenue of publisher k for user type $\bar{c}_k$ when the incentive scheme is g and other publishers' marginal user types are $\bar{\mathbf{c}}_{-k}$. Also, denote $p(\omega, y|c_j)$ as the conditional probability for a user with opportunity cost c_j for publisher j to have touchpoint sequence ω and conversion indicator y.

The scheme $\bar{g}$ is constructed as follows:

1. For publisher j, touchpoint sequence ω_0, and conversion indicator y, set

$$\bar{g}_j(\omega_0, y) = \frac{g_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) + g_j(\omega, y) \cdot p(\omega, y|\bar{c}_j)}{p(\omega_0, y|\bar{c}_j)},$$

   and set $\bar{g}_j(\omega, y) = 0$. That is, compared with the scheme g, $\bar{g}$ shifts the compensation to publisher j from the case (ω, y) to the case (ω_0, y).[36]

2. Otherwise, the compensation is the same as in g. That is, if either $k \neq j$ or $\omega' \neq \omega_0, \omega$ or $y' \neq y$ holds, then $\bar{g}_k(\omega', y') = g_k(\omega', y')$.

The proof that $\bar{g}$ results in a higher payoff consists of two steps. First, it will be shown that the marginal user for any publisher k in equilibrium under $\bar{g}$ remains $\bar{c}_k$. Second, it will be demonstrated that the advertising expenditure under $\bar{g}$ is lower. Combining the two steps, it can be concluded that the same set of consumers are shown the advertisement, so the conversion events would be identical, but the advertising expenditure under $\bar{g}$ is lower, making $\bar{g}$ a more profitable scheme.

For the first step, to show that the marginal user for any publisher k under $\bar{g}$ is still $\bar{c}_k$, it suffices to show the counterpart of (15) also holds under $\bar{g}$, that is,

$$\bar{R}_k(\bar{c}_k; \bar{\mathbf{c}}_{-k}) = \bar{c}_k,$$

[36] If the construction $\bar{g}_j$ violates the condition that $\bar{g}_j(\omega_0, 1) \geq \bar{g}_j(\omega_0, 0)$, that is, $\bar{g}$ is not in the feasible set, one can similarly construct a new scheme $\bar{\bar{g}}$ that shifts the compensation to publisher j from the case $(\omega_0, 0)$ to the case $(\omega_0, 1)$, and then $\bar{\bar{g}}$ is in the feasible set. Since the conversion outcome is independent of opportunity costs, it can be shown that $\bar{g}$ and $\bar{\bar{g}}$ are equivalent, so the proof below implies that $\bar{\bar{g}}$ that yields a higher payoff than g for the advertiser.

where $\bar{R}_k(\bar{c}_k; \bar{\mathbf{c}}_{-k})$ is the expected revenue of publisher k for user type $\bar{c}_k$ when the incentive scheme is $\bar{g}$ and other publishers' marginal users are $\bar{\mathbf{c}}_{-k}$. For publisher $k \neq j$, since $\bar{g}_k = g_k$ by construction, we have $\bar{R}_k = R_k$ and thus the equation holds. For publisher $k = j$,

$$R_j(\bar{c}_j; \bar{\mathbf{c}}_{-j}) = \sum_{(\omega', y'): j \in \omega'} g_j(\omega', y') \cdot p(\omega, y|\bar{c}_j),$$

and the expressions of $\bar{R}_j(\bar{c}_j; \bar{\mathbf{c}}_{-j})$ only differ at (ω_0, y) and (ω, y), so we can write

$$\begin{aligned}
&\bar{R}_j(\bar{c}_j; \bar{\mathbf{c}}_{-j}) - R_j(\bar{c}_j; \bar{\mathbf{c}}_{-j}) \\
=&\, \bar{g}_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) + \bar{g}_j(\omega, y) \cdot p(\omega, y|\bar{c}_j) - g_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) - g_j(\omega, y) \cdot p(\omega, y|\bar{c}_j) \\
=&\, \frac{g_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) + g_j(\omega, y) \cdot p(\omega, y|\bar{c}_j)}{p(\omega_0, y|\bar{c}_j)} \cdot p(\omega_0, y|\bar{c}_j) - g_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) - g_j(\omega, y) \cdot p(\omega, y|\bar{c}_j) \\
=&\, 0.
\end{aligned}$$

The second step of the proof focuses on the comparison of advertising expenditures under the g and $\bar{g}$ schemes. First, the advertising expenditures for any publisher $k \neq j$ remain the same under both g and $\bar{g}$. Moving on to publisher j, let $f_j(c_j)$ represent the marginal density of c_j, then the expenditure under g can be expressed as

$$\mathbb{E}[g_j(T, Y)] = \int_{c_j \leq \bar{c}_j} \sum_{(\omega, y): j \in \omega} g_j(\omega', y') \cdot p(\omega', y'|\bar{c}_j) \cdot f_j(c_j)\, \mathrm{d}c_j.$$

Since g and $\bar{g}$ only differ at (ω_0, y) and (ω, y), the difference between the expected values of $\bar{g}_j(T, Y)$ and $g_j(T, Y)$ can be obtained through the following equation:

$$\begin{aligned}
&\mathbb{E}[\bar{g}_j(T, Y)] - \mathbb{E}[g_j(T, Y)] \\
=&\int_{c_j \leq \bar{c}_j} ((\bar{g}_j(\omega_0, y) - g_j(\omega_0, y)) \cdot p(\omega_0, y|\bar{c}_j) + (\bar{g}_j(\omega, y) - g_j(\omega, y)) \cdot p(\omega, y|\bar{c}_j)) \cdot f_j(c_j)\, \mathrm{d}c_j.
\end{aligned}$$

Thus, it suffices to show that for $c_j < \bar{c}_j$, the following holds true:

$$\bar{g}_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) + \bar{g}_j(\omega, y) \cdot p(\omega, y|\bar{c}_j) \leq g_j(\omega_0, y) \cdot p(\omega_0, y|\bar{c}_j) + g_j(\omega, y) \cdot p(\omega, y|\bar{c}_j).$$

Considering that $g_j(\omega, y) > 0$ and every component in this inequality is nonnegative, this inequality holds true if and only if the following condition is met:

$$\frac{p(\omega_0, y|c_j)}{p(\omega, y|c_j)} \leq \frac{p(\omega_0, y|\bar{c}_j)}{p(\omega, y|\bar{c}_j)}. \tag{16}$$

Now, let us revisit the assumption made in Proposition 3.1 where the conversion rate is not dependent on the opportunity cost vector c. With this assumption, the expansion of $p(\omega, y|c_j)$ can be written as follows:

$$
\begin{aligned}
& p(\omega, y|c_j) \\
= & \mathbb{P}(Y = y|T = \omega) \cdot \mathbb{P}(T = \omega|c_j) \\
= & \mathbb{P}(Y = y|T = \omega) \cdot \int \mathbb{1}(T(c) = \omega) \cdot f_{-j}(c_{-j}|c_j) \, \mathrm{d}c_{-j} \\
= & \mathbb{P}(Y = y|T = \omega) \cdot \int \prod_{k \neq j} (\mathbb{1}(k \in \omega, c_k \leq \bar{c}_k) + \mathbb{1}(k \notin \omega, c_k > \bar{c}_k)) \cdot f_{-j}(c_{-j}|c_j) \, \mathrm{d}c_{-j}.
\end{aligned}
$$

The last equality holds since there are two potential cases for any publisher $k \neq j$: either $k \in \omega$, or $k \notin \omega$. The condition $T(c) = \omega$ is satisfied if and only if the former publishers display the advertisement ($c_k \leq \bar{c}_k$ for $k \in \omega$) and the latter publishers do not ($c_k > \bar{c}_k$ for $k \notin \omega$).

Given that $\omega_0 = (j) \subset \omega$, we can denote $S_1 = \{k : k \in \omega, k \neq j\}$, and $S_2 = \{k : k \notin \omega\}$. Note that $S_1 \neq \emptyset$. We can write

$$
p(\omega_0, y|c_j) = \mathbb{P}_{Y|T}(Y = y|T = \omega_0) \cdot \int_{c_{S_1} > \bar{c}_{S_1}, c_{S_2} > \bar{c}_{S_2}} f_{-j}(c_{S_1}, c_{S_2}|c_j) \, \mathrm{d}c_{S_1} \, \mathrm{d}c_{S_2},
$$

$$
p(\omega, y|c_j) = \mathbb{P}_{Y|T}(Y = y|T = \omega) \cdot \int_{c_{S_1} \leq \bar{c}_{S_1}, c_{S_2} > \bar{c}_{S_2}} f_{-j}(c_{S_1}, c_{S_2}|c_j) \, \mathrm{d}c_{S_1} \, \mathrm{d}c_{S_2}.
$$

Given the log-supermodularity of f, and the condition $c_j < \bar{c}j$, we can formulate

$$
\frac{p(\omega_0, y|c_j)}{p(\omega_0, y|\bar{c}_j)} \leq \frac{\int_{\bar{c}_{S_2}}^{\infty} f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j) \, \mathrm{d}c_{S_2}}{\int_{\bar{c}_{S_2}}^{\infty} f_{-j}(\bar{c}_{S_1}, c_{S_2}|\bar{c}_j) \, \mathrm{d}c_{S_2}},
$$

$$
\frac{p(\omega, y|c_j)}{p(\omega, y|\bar{c}_j)} \geq \frac{\int_{\bar{c}_{S_2}}^{\infty} f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j) \, \mathrm{d}c_{S_2}}{\int_{\bar{c}_{S_2}}^{\infty} f_{-j}(\bar{c}_{S_1}, c_{S_2}|\bar{c}_j) \, \mathrm{d}c_{S_2}}.
$$

These inequalities imply that (16) holds true, and they are strict if f is strictly log-supermodular.

C.5.2 Adaptations in the Proof of Proposition 3.2

The key techniques employed in the proof of Proposition 3.2 are the same as previously discussed.

The first step of the proof is to demonstrate that when $h(\omega, c)$ increases in c for a fixed ω, then the optimal scheme has $g_j(\omega, 0) = 0$.

The proof leverages the Fortuin–Kasteleyn–Ginibre (FKG) inequality, which is presented below.

Lemma C.1 (FKG inequality)**.** *Suppose that a probability density function μ of a random* *vector X exhibit affiliation. Then for any two nondecreasing functions $\phi_1(x)$ and $\phi_2(x)$, the following inequality holds: $\mathbb{E}[\phi_1(X)\cdot\phi_2(X)] \geq \mathbb{E}[\phi_1(X)]\cdot\mathbb{E}[\phi_2(X)]$.*

Suppose we have $g_j(\omega,0) > 0$ for touchpoint sequence ω, then we can construct an alternative scheme $\bar{g}$ that $\bar{g}_j(\omega,0) = 0$ and

$$\bar{g}_j(\omega,1) = \frac{g_j(\omega_0,0)\cdot p(\omega,0|\bar{c}_j) + g_j(\omega,1)\cdot p(\omega,1|\bar{c}_j)}{p(\omega,1|\bar{c}_j)}.$$

Proceeding as the same procedure, it suffices to show that for $c_j < \bar{c}_j$,

$$\frac{p(\omega,1|c_j)}{p(\omega,0|c_j)} \leq \frac{p(\omega,1|\bar{c}_j)}{p(\omega,0|\bar{c}_j)},$$

which is equivalent to

$$\mathbb{P}_{c_{-j}|c_j,T}(Y=1|c_j,T=\omega) \leq \mathbb{P}_{c_{-j}|c_j,T}(Y=1|\bar{c}_j,T=\omega).$$

Expanding yields

$$\mathbb{P}_{c_{-j}|c_j,T}(Y=1|c_j,T=\omega) = \mathbb{E}_{c_{-j}|c_j,T}[h(\omega,c_j,c_{-j})] = \int h(\omega,c_j,c_{-j})\cdot f_{-j}(c_{-j}|c_j,T=\omega)\,\mathrm{d}c_{-j},$$

where

$$f_{-j}(c_{-j}|c_j,T=\omega) = \frac{\prod_{k\neq j}\left(\mathbb{1}(k\in\omega, c_k\leq\bar{c}_k) + \mathbb{1}(k\notin\omega, c_k>\bar{c}_k)\right)\cdot f_{-j}(c_{-j}|c_j)}{\mathbb{P}(T=\omega|c_j)}.$$

Then, we can express

$$\begin{aligned}
&\mathbb{P}_{c_{-j}|c_j,T}(Y=1|\bar{c}_j,T=\omega)\\
=&\int h(\omega,\bar{c}_j,c_{-j})\cdot f_{-j}(c_{-j}|\bar{c}_j,T=\omega)\,\mathrm{d}c_{-j}\\
\geq&\int h(\omega,c_j,c_{-j})\cdot f_{-j}(c_{-j}|\bar{c}_j,T=\omega)\,\mathrm{d}c_{-j}\\
=&\int h(\omega,c_j,c_{-j})\cdot\frac{f_{-j}(c_{-j}|\bar{c}_j,T=\omega)}{f_{-j}(c_{-j}|c_j,T=\omega)}\cdot f_{-j}(c_{-j}|c_j,T=\omega)\,\mathrm{d}c_{-j}.
\end{aligned}$$

By log-supermodularity of f, the probability density function $f_{-j}(c_{-j}|c_j,T=\omega)$ is log-supermodular in c_{-j} and $\dfrac{f_{-j}(c_{-j}|\bar{c}_j,T=\omega)}{f_{-j}(c_{-j}|c_j,T=\omega)}$ increases in c_{-j}. Applying Lemma C.1, we obtain

$$\begin{aligned}
&\int h(\omega,c_j,c_{-j})\cdot\frac{f_{-j}(c_{-j}|\bar{c}_j,T=\omega)}{f_{-j}(c_{-j}|c_j,T=\omega)}\cdot f_{-j}(c_{-j}|c_j,T=\omega)\,\mathrm{d}c_{-j}\\
\geq&\left(\int h(\omega,c_j,c_{-j})\cdot f_{-j}(c_{-j}|c_j,T=\omega)\,\mathrm{d}c_{-j}\right)\cdot\left(\int\frac{f_{-j}(c_{-j}|\bar{c}_j,T=\omega)}{f_{-j}(c_{-j}|c_j,T=\omega)}\cdot f_{-j}(c_{-j}|c_j,T=\omega)\,\mathrm{d}c_{-j}\right)
\end{aligned}$$

$$= \mathbb{P}_{c_{-j}|c_j,T}(Y = 1|c_j, T = \omega) \cdot 1,$$

thus confirming the claim $\mathbb{P}_{c_{-j}|c_j,T}(Y = 1|\bar{c}_j, T = \omega) \geq \mathbb{P}_{c_{-j}|c_j,T}(Y = 1|c_j, T = \omega)$.

The second step of the proof proceeds also by contradiction. With the optimal scheme setting $g_j(\omega, 0) = 0$, we can shift our focus to $g_j(\omega, 1)$. Akin to the procedure laid out in Proposition 3.1, we construct an analogous scheme $\bar{g}$ which yields a greater payoff. The proof thus amounts to showing (16) with $y = 1$.

The proof leverages the Ahlswede–Daykin inequality, which is presented below.

Lemma C.2 (Ahlswede–Daykin inequality)**.** *Suppose that $\phi_1, \phi_2, \phi_3, \phi_4$ are nonnegative integrable functions on $\mathbb{R}^d$, such that for any $x_1, x_2 \in \mathbb{R}^d$, $\phi_1(x_1) \cdot \phi_2(x_2) \leq \phi_3(x_1 \vee x_2) \cdot \phi_4(x_1 \wedge x_2)$, then it follows that $\int \phi_1(x)\,\mathrm{d}x \cdot \int \phi_2(x)\,\mathrm{d}x \leq \int \phi_3(x)\,\mathrm{d}x \cdot \int \phi_4(x)\,\mathrm{d}x$.*

Expanding $p(\omega, y|c_j)$, we get

$$\begin{aligned}
&p(\omega, y|c_j) \\
&= \int h(\omega, c) \cdot \mathbb{1}(T(c) = \omega) \cdot f_{-j}(c_{-j}|c_j)\,\mathrm{d}c_{-j} \\
&= \int h(\omega, c) \cdot \prod_{k \neq j} (\mathbb{1}(k \in \omega, c_k \leq \bar{c}_k) + \mathbb{1}(k \notin \omega, c_k > \bar{c}_k)) \cdot f_{-j}(c_{-j}|c_j)\,\mathrm{d}c_{-j}.
\end{aligned}$$

Then, we have

$$p(\omega_0, y|c_j) = \int_{c_{S_1} > \bar{c}_{S_1}, c_{S_2} > \bar{c}_{S_2}} h(\omega_0, c_j, c_{S_1}, c_{S_2}) \cdot f_{-j}(c_{S_1}, c_{S_2}|c_j)\,\mathrm{d}c_{S_1}\,\mathrm{d}c_{S_2},$$

$$p(\omega, y|c_j) = \int_{c_{S_1} \leq \bar{c}_{S_1}, c_{S_2} > \bar{c}_{S_2}} h(\omega, c_j, c_{S_1}, c_{S_2}) \cdot f_{-j}(c_{S_1}, c_{S_2}|c_j)\,\mathrm{d}c_{S_1}\,\mathrm{d}c_{S_2}.$$

Given the log-supermodularity of f and ρ, and considering $c_j < \bar{c}_j$, we have

$$\frac{p(\omega_0, y|c_j)}{p(\omega_0, y|\bar{c}_j)} \leq \frac{\int_{\bar{c}_{S_2}}^{\infty} h(\omega_0, c_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j)\,\mathrm{d}c_{S_2}}{\int_{\bar{c}_{S_2}}^{\infty} h(\omega_0, \bar{c}_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|\bar{c}_j)\,\mathrm{d}c_{S_2}} \equiv A, ,$$

$$\frac{p(\omega, y|c_j)}{p(\omega, y|\bar{c}_j)} \geq \frac{\int_{\bar{c}_{S_2}}^{\infty} h(\omega, c_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j)\,\mathrm{d}c_{S_2}}{\int_{\bar{c}_{S_2}}^{\infty} h(\omega, \bar{c}_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|\bar{c}_j)\,\mathrm{d}c_{S_2}} \equiv B,$$

with the goal to show that $A \leq B$. To utilize Lemma C.2, define

$$\begin{aligned}
\phi_1(c_{S_2}) &= h(\omega_0, c_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j) \cdot \mathbb{1}(c_{S_2} \geq \bar{c}_{S_2}), \\
\phi_2(c_{S_2}) &= h(\omega, \bar{c}_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|\bar{c}_j) \cdot \mathbb{1}(c_{S_2} \geq \bar{c}_{S_2}), \\
\phi_3(c_{S_2}) &= h(\omega_0, \bar{c}_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|\bar{c}_j) \cdot \mathbb{1}(c_{S_2} \geq \bar{c}_{S_2}), \\
\phi_4(c_{S_2}) &= h(\omega, c_j, \bar{c}_{S_1}, c_{S_2}) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j) \cdot \mathbb{1}(c_{S_2} \geq \bar{c}_{S_2}).
\end{aligned}$$

Then for any c_{S_2}, c'_{S_2}, we have the following:

1. Log-supermodularity of $h(\omega, -c)$ gives us

$$h(\omega_0, c_j, \bar{c}_{S_1}, c_{S_2}) \cdot h(\omega, \bar{c}_j, \bar{c}_{S_1}, c'_{S_2}) \leq h(\omega_0, \bar{c}_j, \bar{c}_{S_1}, c_{S_2} \vee c'_{S_2}) \cdot h(\omega, c_j, \bar{c}_{S_1}, c_{S_2} \wedge c'_{S_2}).$$

2. Log-supermodularity of f gives us

$$f_{-j}(\bar{c}_{S_1}, c_{S_2}|c_j) \cdot f_{-j}(\bar{c}_{S_1}, c'_{S_2}|\bar{c}_j) \leq f_{-j}(\bar{c}_{S_1}, c_{S_2} \vee c'_{S_2}|\bar{c}_j) \cdot f_{-j}(\bar{c}_{S_1}, c_{S_2} \wedge c'_{S_2}|c_j).$$

3. The definition of the join and the meet gives us

$$\mathbb{1}(c_{S_2} \geq \bar{c}_{S_2}) \cdot \mathbb{1}(c'_{S_2} \geq \bar{c}_{S_2}) = \mathbb{1}(c_{S_2} \vee c'_{S_2} \geq \bar{c}_{S_2}) \cdot \mathbb{1}(c_{S_2} \wedge c'_{S_2} \geq \bar{c}_{S_2}).$$

From these deductions, we arrive at

$$\phi_1(c_{S_2}) \cdot \phi_2(c'_{S_2}) \leq \phi_3(c_{S_2} \vee c'_{S_2}) \cdot \phi_4(c_{S_2} \wedge c'_{S_2}).$$

Applying Lemma C.2 confirms that $A \leq B$ holds, and thus, (16) holds, with the inequalities strict if the conditions are strict.

C.6 Proof of Proposition B.1

Similar to the proof of Proposition 2.1, the key is to evaluate $\dfrac{\partial Q_j}{\partial a_{ij}^{(k)}}$. Note that comparing with not displaying any advertisements to user i, if publisher j display advertiser k's advertisement, only the touchpoint sequence for advertiser k will be affected. Moreover, the model assumes that this user's conversion on other advertisers does not depend on the touchpoint sequence for advertiser k, so their bids will not be affected. Thus, this display only has an direct effect on advertiser k's attributed conversion rates and future bids. Thus, similar to the proof of Proposition 2.1, we have

$$\tilde{g}_j^{(k)}(\omega, y) = b_j^{(k)*} + \delta \cdot \frac{\partial V_j(\mathbf{b}_j^*)}{\partial b_j^{(k)*}} \cdot \frac{\partial \psi_j^{(k)}(b_j^{(k)*}, r_j^{(k)})}{\partial \pi_{\omega,y}^{(k)}}, \tag{17}$$

and the rest of the proof is similar.

D Static Equivalent Examples

In this appendix, I provide several simplified examples of static equivalents.

D.1 Equivalence of Pay-per-conversion and Target Cost-per-action Bidding

When we have $J = 1$, meaning the advertiser engages with just one publisher for their advertising campaign, the game simplifies into a publisher's dynamic decision problem. Omit the publisher subscript and let $\omega = 1$ denote the event of the publisher displaying the advertisement. The advertiser's subsequent bid for an impression, $b' = \psi(r)$, is contingent on the present period's conversion rate r.[37] The following result shows that the incentive for the publisher, created by the attribution algorithms, is a mixture of pay-per-impression and pay-per-conversion schemes. Moreover, when the advertiser sets bids linearly in the conversion rates, the static equivalent is a pay-per-conversion scheme.

Corollary D.1. *Suppose that there is $J = 1$ publisher, and the advertiser's next period's bid for an impression is $b' = \psi(r)$, with r being the conversion rate of the current period. Suppose also that the conversion rate is 0 if the advertisement is not displayed. Let (a^*, b^*) be an oblivious equilibrium of the game, then the static equivalent*

$$\tilde{g}(y) = (1 - \eta) \cdot b^* + \eta \cdot v^* \cdot \mathbb{1}(y = 1),$$

a mixture of a pay-per-impression scheme and a pay-per-conversion scheme. The mixture weight $\eta = \delta \cdot \frac{r \cdot \psi'(r)}{b^*}$ is the product of publisher j's discount factor and the elasticity of ψ at the stationary conversion rate r and $v^* = \frac{b^*}{r}$ is the effective payment per conversion. Moreover, when $\delta \to 1$ and $\psi(r) = v^* \cdot r$, that is, the advertiser's future bid is linear in the current conversion rate, then

$$\tilde{g}(y) = v^* \cdot \mathbb{1}(y = 1),$$

that is, the static equivalent is a pay-per-conversion with a bid v^*.

As a real-world example, Google offers two bidding mechanisms to advertisers: (i) pay-per-conversion: the advertiser places a bid of v for conversions and compensates Google this amount upon user conversion; (ii) target cost-per-action bidding: the advertiser bids b for impressions and pays Google this amount upon the occurrence of an advertisement impression, with the bid b is dynamically adjusted. The advertiser assigns a CPA target v and establishes an automated bidding algorithm on Google. In each advertising auction, the algorithm uses historical data to compute a conversion rate r for each impression, subsequently placing a bid of $v \cdot r$ for an impression. Algorithm (i) corresponds to a static PPA model in the static game, while algorithm (ii) aligns with a dynamic pay-per-impression bidding algorithm in the dynamic game, as described in the corollary. The

[37] In this special case, there is no need for attribution, that is, every conversion is attributed to the publisher.

corollary highlights that these two algorithms essentially exhibit equivalent behavior, as both the advertisement display decisions and the payoffs for the advertiser and publisher are the same under each mechanism. For algorithm (ii), even though the bid is a constant b for all users, because of the dynamic incentives introduced by the algorithm, the publisher still has the incentive to display the advertisement to users with higher conversion rates, as in algorithm (i). The dynamic incentives work as follows: from the publisher's perspective, displaying the advertisement to a user with a higher conversion rate could increase the advertiser's future bid, thereby enhancing the publisher's incentives to show the advertisement to users with higher conversion rates.

E Assumptions to Ensure Thresholding Strategies

In this appendix, I provide a set of sufficient conditions that, if satisfied, can ensure that any Bayesian Nash equilibrium is a thresholding strategy profile. One may think that a sufficient condition is publisher j's revenue $R_j(c_j; \mathbf{a}_{-j})$'s derivative with regard to c_j is less than 1. However, this revenue is a function of other publishers' strategies $\mathbf{a}_{-j}$, which are endogenous variables, thereby making this assumption challenging to verify. To circumvent this complication, I propose the following set of assumptions based on the primitives.

Assumption A1 (Bounded Derivatives of Conditional Opportunity Cost Distribution)**.** There exists $L_1 > 0$ such that for any publisher j and measurable partition $(\mathcal{C}_{-j}^{(k)})_k$ of $\mathbb{R}^{J-1}$, the following condition holds:

$$\sum_k \left| \frac{\partial \, \mathbb{P}_{c_{-j}|c_j}(c_{-j} \in \mathcal{C}_{-j}^{(k)} | c_j)}{\partial c_j} \right| \leq L_1.$$

In essence, Assumption A1 postulates that the conditional distribution of c_{-j} given c_j, does not exhibit excessive sensitivity to c_j.[38]

Assumption A2 (Bounded Derivatives of Conditional Conversion Rate)**.** There exists $L_2 > 0$ such that for any publisher j, touchpoint sequence ω and measurable partition $(\mathcal{C}_{-j}^{(k)})_k$ of $\mathbb{R}^{J-1}$, the following condition holds:

$$\sum_k \left| \frac{\partial \, \mathbb{P}_{c_{-j}|c_j}(c_{-j} \in \mathcal{C}_{-j}^{(k)}, Y = 1 | c_j)}{\partial c_j} \right| \leq L_2.$$

[38] For example, for $J = 2$, suppose $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ admits a bivariate normal distribution with mean $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ and covariance $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. In such a case, it can be shown that Assumption A1 holds with $L_1 = \dfrac{|\rho|}{\sqrt{2\pi}\min\{\sigma_1, \sigma_2\}}$.

Assumption A2 implies that the conversion rate does not exhibit excessive sensitivity to the opportunity costs.[39]

Take into account that the incentives provided to publishers are subject to an upper limit of M. The following lemma indicates that, under these assumptions, and provided the corresponding bounds satisfy a further restriction, we can ensure any Bayesian Nash equilibrium consists of thresholding strategies.

Lemma E.1. *If Assumptions A1, A2, and $M \cdot (L_1 + L_2) < 1$ holds, then in a Bayesian Nash equilibrium $\boldsymbol{a}^*$, each publisher implements a thresholding strategy.*

Proof. Fix other publishers' strategies $\mathbf{a}_{-j}$. Note that $\mathbf{a}_{-j}$ depends on c_{-j}. Denote $\mathcal{C}_{-j}^{(\omega)}$ as the set of c_{-j} such that if publisher j display the advertisement, and other publishers' display decision is $\mathbf{a}_{-j}(c_{-j})$, the user's touchpoint sequence is ω. More formally,

$$\mathcal{C}_{-j}^{(\omega)} = \{c_{-j} : \Omega\left(H; 1, \mathbf{a}_{-j}(c_{-j})\right) = \omega\},$$

where $\Omega(H; \mathbf{a})$ maps browsing path H and display decisions $\mathbf{a}$ to a sequence of touchpoints. We can rewrite

$$\begin{aligned}
&R_j(c_j; \mathbf{a}_{-j}) \\
=&\mathbb{E}_{c_{-j},Y|c_j}\left[g_j(T, Y)|c_j\right] \\
=&\mathbb{E}_{c_{-j}|c_j}\left[g_j(T, 1)\cdot \mathbb{1}(Y=1) + g_j(T,0)\cdot(1-\mathbb{1}(Y=1))|c_j\right] \\
=&\mathbb{E}_{c_{-j}|c_j}\left[g_j(T, 0) + (g_j(T,1)-g_j(T,0))\cdot \mathbb{1}(Y=1)|c_j\right] \\
=&\sum_{\omega}\Big(g_j(\omega,0)\cdot \mathbb{P}_{c_{-j}|c_j}(c_{-j}\in \mathcal{C}_{-j}^{(\omega)}|c_j) + (g_j(\omega,1)-g_j(\omega,0))\cdot \mathbb{P}_{c_{-j}|c_j}(c_{-j}\in \mathcal{C}_{-j}^{(\omega)}, Y=1|c_j)\Big).
\end{aligned}$$

Then,

$$\begin{aligned}
&\frac{\partial R_j(c_j; \mathbf{a}_{-j})}{\partial c_j} \\
=&\sum_{\omega}\left(g_j(\omega,0)\cdot \frac{\partial\, \mathbb{P}_{c_{-j}|c_j}(c_{-j}\in \mathcal{C}_{-j}^{(\omega)}|c_j)}{\partial c_j} + (g_j(\omega,1)-g_j(\omega,0))\cdot \frac{\partial\, \mathbb{P}_{c_{-j}|c_j}(c_{-j}\in \mathcal{C}_{-j}^{(\omega)}, Y=1|c_j)}{\partial c_j}\right) \\
\leq& M\cdot \sum_{\omega}\left|\frac{\partial\, \mathbb{P}_{c_{-j}|c_j}(c_{-j}\in \mathcal{C}_{-j}^{(\omega)}|c_j)}{\partial c_j}\right| + M\cdot\sum_{\omega}\left|\frac{\partial\, \mathbb{P}_{c_{-j}|c_j}(c_{-j}\in \mathcal{C}_{-j}^{(\omega)}, Y=1|c_j)}{\partial c_j}\right| \\
\leq& M\cdot L_1 + M\cdot L_2
\end{aligned}$$

[39] For example, for $J = 2$, suppose $h(\omega, c) = \Phi(\alpha_\omega + \sum_{j=1}^{2}\lambda_j \cdot c_j)$ and $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$ admits a bivariate normal distribution with mean $\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$, covariance $\begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}$. In such a case, it can be shown that Assumption A2 holds with $L_2 = \frac{1}{\sqrt{2\pi}}\max\left\{\sqrt{\lambda_1^2 + \frac{\rho^2}{\sigma_1^2(1-\rho^2)}}, \sqrt{\lambda_2^2 + \frac{\rho^2}{\sigma_2^2(1-\rho^2)}}\right\}$.

<1.

The last inequality shows that the assumptions guarantee that the derivative of publisher j's revenue $R_j(c_j;\mathbf{a}_{-j})$ with respect to c_j is less than 1, indicating that $R_j(c_j;\mathbf{a}_{-j})-c_j$ decreases in c_j. Publisher j will choose to display the advertisement if and only if $R_j(c_j;\mathbf{a}_{-j})$ is no less than c_j, which holds if and only if c_j is sufficiently small. □

F Optimal Attribution Design

In this appendix, I show that the optimal incentive defined in Proposition 3.2 is the static equivalent of an attribution algorithm Ψ^{O}.

Denote the optimal incentive scheme as g^*. For a given publisher j, touchpoint sequence ω and conversion indicator y, $g_j^*(\omega, y)$ is non-negative when $\omega = (j)$ and $y = 1$, otherwise, $g_j^*(\omega, y) = 0$. In the former scenario, denote $v_j^* \equiv g_j^*(\omega, y)$.

The algorithm Ψ^{O} is constructed as follows. In the attribution step, when a touchpoint sequence includes multiple publishers, the algorithm assigns no credit to any publisher involved. Thus, it is analogous to a rule-based attribution algorithm with weight function $\chi(j,\omega)$, where $\chi(j,\omega) = 1$ if $\omega = (j)$ and $\chi(j,\omega) = 0$ otherwise. Subsequently, the algorithm calculates the attributed conversion rates as in equation (2). In the bid adjustment step, the subsequent period's bid b_j' for publisher j is set as $b_j' = v_j^* \cdot r_j$, effectively multiplying publisher j's attributed conversion rate with the compensation in the optimal incentives.[40] The following Proposition is a direct corollary to Proposition 2.2.

Proposition F.1. *Suppose $(\boldsymbol{a}^*, b^*)$ is an oblivious equilibrium of the dynamic game $G(\Psi^O)$. Suppose also that $\delta \to 1$, then the static equivalent of Ψ^O, denoted as $\tilde{g}^O$, is identical to g^*.*

Proof. Since the proposition is a special case of Proposition 2.2, it suffices to plug in the values. First,

$$\frac{\partial \psi_j(b_j, r_j)}{\partial b_j} = 0,$$

and we need to verify the following inequality holds for any sequence of touchpoints ω and conversion indicator y,

$$-\frac{\partial \psi_j(b_j, r_j)}{\partial \pi_{\omega,y}} \cdot \frac{\sum_{\omega: j\in\omega} \pi_\omega}{\psi_j(b_j, r_j)} \leq 1.$$

[40] These results remain valid if we extend this algorithm such that b_j' is a weighted average of $v_j^* \cdot r_j$ and b_j. Specifically, for a fixed weight $w_j \in [0,1]$, the subsequent period's bid $b_j' = w_j \cdot v_j^* \cdot r_j + (1-w_j)b_j$.

The left-hand-side equals

$$-\frac{v_j^* \cdot (\mathbb{1}(\omega = (j), y = 1) - r_j)}{\sum_{\omega: j\in\omega} \pi_\omega} \cdot \frac{\sum_{\omega: j\in\omega} \pi_\omega}{v_j^* \cdot r_j} = \frac{r_j - \mathbb{1}(\omega = (j), y = 1)}{r_j} \leq 1,$$

so the inequality holds.

Then, evaluating η_j^* in Proposition 2.2 yields

$$\eta_j^* = \delta \cdot (1 - \delta \cdot 0)^{-1} \cdot v_j^* \cdot \frac{r_j}{v_j^* \cdot r_j} = 1,$$

and since $\chi(j, \omega) = 1$ if $\omega = (j)$ and $\chi(j, \omega) = 0$ otherwise, the static equivalent is simplified to

$$\tilde{g}_j(\omega) = v_j^* \cdot \mathbb{1}(\omega = (j), y = 1),$$

which is identical to the optimal incentive. □

The intuition of the proposition is that credit-giving in the dynamic setting mirrors compensation in the static setting. Thus, by giving no credit to any publishers when there are multiple in the touchpoint sequence, the incentives created by the algorithm resemble the optimal incentive.

G Validation Experiment

In this appendix, I outline the design of a planned validation experiment to provide empirical evidence for the effectiveness of the optimal incentive scheme. The goal of the experiment is to validate that the optimal incentive scheme can lead to a substantial increase in ROI compared to the optimal static per-impression bidding strategy.

To conduct the validation experiment, I collaborate with another advertiser who has $M = 6$ user segments. Prior to the experiment, I will conduct a pilot study with a design similar to the experiment described in Section 4. In the pilot study, I will submit randomized bids for this advertiser, allowing me to estimate a structural model for each of the user segments. Based on these estimates, I will calculate the optimal static per-impression bid and the bid under the optimal incentive scheme for each segment. I will also compute the counterfactual ROIs for both schemes, which will be validated in the main experiment.

The main experiment is planned to span eight weeks. The six user segments will be randomly divided into two groups: three segments in the treatment group and three segments in the control group. During the first two weeks of the experiment, both the treatment and control groups will adopt the optimal static per-impression bid. Starting from the third week, the treatment group will switch to the optimal incentive scheme, while the control group will continue with the optimal static per-impression bid. The primary outcome of interest is the weekly ROI for each treatment condition.

To assess the validity of the structural model, I will compare the observed weekly ROI for each treatment condition with the values predicted by the model. Formally, the hypothesis to be tested is that the optimal incentive scheme leads to an ROI that is significantly higher than the ROI achieved by the optimal static per-impression bidding strategy.

The random assignment of user segments to treatment and control groups ensures that any differences in ROI between the two groups can be attributed to the different bidding strategies rather than inherent differences in the user segments. The staggered implementation of the optimal incentive scheme allows for a clear comparison of the two strategies while controlling for potential time-related confounds.

To enhance the robustness of the findings, I will conduct additional analyses, such as testing for heterogeneous treatment effects across different user segments and exploring the sensitivity of the results to different model specifications. I will also discuss the practical implications of the findings for advertisers and publishers, highlighting the potential improvement in ROI from adopting the optimal incentive scheme.

The proposed validation experiment will provide empirical evidence for the effectiveness of the optimal incentive scheme. By comparing the observed ROI with the values predicted by the structural model, this experiment will demonstrate the practical applicability of the proposed solution and strengthen the credibility of the findings presented in this paper.