# Dual Process Utility Theory:
# A Model of Decisions Under Risk and Over Time

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ABSTRACT. The Discounted Expected Utility model has been a major workhorse for analyzing individual behavior for over half a century. However, it cannot account for evidence that risk interacts with time preference, that time interacts with risk preference, that many people are averse to timing risk and do not discount the future exponentially, that discounting depends on the magnitude of outcomes, that risk preferences are not time preferences, and that risk and time preferences are correlated with cognitive ability. Here we address these issues in a decision model based on the interaction of an affective and a reflective valuation process. The resulting Dual Process Utility theory provides a unified approach to modeling risk preference, time preference, and interactions between risk and time preferences. It also provides a unification of models based on a rational economic agent, models based on prospect theory or rank-dependent utility, and dual system models of decision making.

*Keywords:* Risk preference; Time preference; Cognitive Ability; System 1; System 2
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# 1. INTRODUCTION

Many decisions in life involve uncertain outcomes that materialize at different points in time. For example, the struggle to kick an addiction involves a tradeoff between short term gratification and an increased risk of future health problems. Saving for retirement involves consideration of preferences for immediate consumption and uncertainty about future income. Whether to pursue a long-term project involves consideration of the time the project is expected to take and the likelihood of project success. The decision to purchase a warranty on a television or appliance involves a higher immediate cost, but reduced product breakdown risk. The decision to take a 'buy-it-now' option on eBay or wait until the auction ends for the chance of a better deal, the decision to purchase a laptop today or wait for a potential Black Friday sale, and the decision to take out a mortgage on a home or wait for a possibly lower interest rate each involve a tradeoff between a certain, immediate payoff and a risky, delayed payoff.

As these examples illustrate, decisions often involve both risk and time delays. Yet the domains of risk and time have traditionally been studied separately. In cases where risk and time preferences are both considered, the discounted expected utility (DEU) model remains a major workhorse for analyzing individual behavior. There are, however, a variety of important shortcomings of DEU. For instance, it implies that introducing risk has no effect on time preference, that introducing delays has no effect on risk preference, that risk and time preferences are generated by the same utility function, that people are risk-seeking toward lotteries over uncertain payment dates, that people discount the future exponentially, and that discounting does not depend on the magnitude of outcomes. All of these predictions have been contradicted by experimental evidence.

In this paper, we introduce a dual process model of choices under risk and over time that naturally resolves each of these limitations of the DEU model. The model generalizes both rank-dependent utility theory (Quiggin, 1982), and Mukherjee's (2010) dual system model of choices under risk to develop a unified model that accounts for attitudes toward risk, attitudes toward time, and a variety of interaction effects between risk and time preferences. The proposed model also provides a unification of three classes of decision models – rank-dependent utility theory, expected utility, and dual process (or dual selves) theories. We refer to the model developed here as Dual Process Utility (DPU) theory.

The DPU theory introduces a new parameter into the analysis of economic decision models which represents the decision maker's 'cognitive type' or 'cognitive ability'. Essentially, an agent's 'cognitive type' identifies whether a person naturally engages in more intuitive and feeling-based processing or relies on more analytical and calculation-based processing. While cognitive ability has been found to correlate with a wide variety of economic behaviors including risk and time preferences (e.g., Frederick, 2005; Burks et al., 2009), saving behavior (Ballinger et al., 2011), strategic sophistication (Carpenter et al., 2013), and efficiency in experimental asset markets (Corgnet et al., 2015), it appears nowhere in the conventional economic models of individual choice.

In addition to introducing the DPU model, we provide DPU with a strong theoretical foundation, motivated by plausible psychological assumptions regarding the properties of dual systems in decision making, as well as by a simple axiomatic approach in which the convex combination functional form of DPU and the existence and uniqueness of the parameter representing the decision maker's 'cognitive type' are implied by the axioms.

We subsequently show that DPU predicts a variety of the major empirical findings regarding risk and time preferences. After providing some background in §2, the model is introduced in §3. In §4 we demonstrate that DPU explains two important empirical violations of discounted utility theory (present bias and the magnitude effect). In §5, we show that DPU explains empirically observed interaction effects between risk and time preferences (time affects risk preference, risk affects time preference, subendurance). In §6 - §8, we demonstrate that DPU permits a separation between risk and time preferences, that it predicts a preference for diversifying payoffs across time, and that it predicts risk aversion toward timing risk. In §9 - §11, we consider additional implications and properties of DPU. The behaviors implied by the model are summarized in §12. A simple strategy for parameter elicitation is noted in §13. Related literature is discussed in §14. Concluding remarks are provided in §15.

## 2. BACKGROUND

The study of risk preferences and time preferences, both analytically and empirically, has been the primary focus of research on individual choice for over half a century. However, although expected utility theory was axiomatized by von Neumann and Morgenstern in 1947, and discounted utility theory was axiomatized by Koopmans in 1960, it was not until 1991 when researchers identified remarkable parallels between the major anomalous behaviors across both domains – such as a common ratio effect in choice under risk and a common difference effect in choice over time (Prelec and Loewenstein, 1991). Since the 'common approach' to risk and time preferences pioneered by Prelec and Loewenstein (1991), some models have been developed to explain behaviors across both domains. For instance, models of similarity judgments apply the same cognitive process to explain anomalies under risk and anomalies over time (Rubinstein 1988; Leland 1994; Leland 2002; Rubinstein, 2003). However, this approach does not address the basic question of how risk and time preferences *interact*.

It has been only fairly recently that attention has shifted to explaining interactions between risk and time preferences. This research direction was partially spurred by experimental studies from Keren and Roelofsma (1995) and Baucells and Heukamp (2010) who each observed different and systematic interactions between risk and time preferences. For instance, Keren and Roelofsma (1995) observed that uncertainty induces more patient behavior. Baucells and Heukamp (2010) and Abdellaoui et al. (2011) both observed that time delays induce more risk-taking behavior. Andersen et al. (2011) and Miao and Zhong (2015) observed a preference for diversifying risks across time. Onay and Onculer (2007) and DeJarnette et al. (2015) observed risk-aversion to lotteries over uncertain payment dates. These behaviors are illustrated in Table 1.

An intuitive approach to modeling risky and intertemporal choices is to multiply a time discount function by a probability weighting function by a utility or value function. However, this general approach does not explain the finding that time affects risk preference (see Table 1) since both alternatives in the example by Baucells and Heukamp (2010) are delayed by the same amount (e.g., three months) and so the discount weights cancel when comparing options A and B. This approach also does not explain the finding that risk affects time preference, since both payoffs in the example by Keren and Roelofsma (1995) have the same probability (e.g., 50%), and so the probability weights cancel when comparing options A and B. In addition, this approach does not explain the finding of subendurance in the example by Baucells et al. (2009), since both options have the same payoffs (e.g., €100) and so the utilities cancel when comparing options A and B. It is then not obvious how to model such interaction effects between time delays, probabilities, and payoffs. It may be even less clear how to derive behaviors in the direction observed in experimental studies, or whether the same approach that might explain interaction effects for time delays can also explain interaction effects for probabilities, and payoffs. We will show that such a unified approach to these interaction effects is not only possible, but has a simple and intuitive interpretation.

**Table 1. Choices between Options A and B involving Risk and Time**

| Observation | Option A vs. | Option B |
|---|---|---|
| **Common Ratio Effect*** (Baucells and Heukamp, 2010) | **(9, for sure, now)**<br>(9, with 10%, now) | (12, with 80%, now)<br>**(12, with 8%, now)** |
| **Common Difference Effect**** (Keren and Roelofsma, 1995) | **(100, for sure, now)**<br>(100, for sure, 26 weeks) | (110, for sure, 4 weeks)<br>**(110, for sure, 30 weeks)** |
| **Time affects Risk Preference*** (Baucells and Heukamp, 2010) | **(9, for sure, now)**<br>(9, for sure, 3 months) | (12, with 80%, now)<br>**(12, with 80%, 3 months)** |
| **Risk affects Time Preference**** (Keren and Roelofsma, 1995) | **(100, for sure, now)**<br>(100, with 50%, now) | (110, for sure, 4 weeks)<br>**(110, with 50%, 4 weeks)** |
| **Subendurance*** (Baucells et al., 2009) | **(100, for sure, 1 month)**<br>(5, for sure, 1 month) | (100, with 90%, now)<br>**(5, with 90%, now)** |
| **Diversification across Time***** (Miao and Zhong, 2015) | **If Heads: 100, now**<br>**If Tails: 100, 1 week** | If Heads: 100 now, 100, 1 week<br>If Tails: 0 now, 0, 1 week |
| **Aversion to Timing Risk***** (Onay and Onculer, 2007) | If Heads: 100, 5 weeks<br>If Tails: 100, 15 weeks | **If Heads: 100, 10 weeks**<br>**If Tails: 100, 10 weeks** |

Adapted from Baucells and Heukamp (2012). Complementary probabilities for all options correspond to payoffs of 0. Sources of experimental results are in parentheses. Majority responses of experimental subjects are in bold font. *Currency in Euros; ** Currency in Dutch Guilders; *** Prototypical examples.

In Table 1, the first five examples are adapted from Baucells and Heukamp (2012). The last two examples are prototypical illustrations of a preference for diversification across time and aversion to timing risk with outcomes determined by the toss of a fair coin.

Further research on relations between risk and time preferences was spurred by Halevy (2008) and Saito (2011) who investigated formal equivalences between the Allais paradox and hyperbolic discounting, and by Andreoni and Sprenger (2012) who re-opened a debate on the limitations of DEU.

### 2.1. *Dual Processes in Decision Making*

Recent literature in cognitive science argues that people do not have a single mental processing system, but rather have two families of cognitive processes. Qualitatively similar distinctions have been made by many authors. Stanovich and West (2000), and Kahneman and Frederick (2002) label these families neutrally as System 1 processes and System 2 processes where System 1 includes automatic, intuitive and affective processes and System 2 includes more deliberative, logical, and reflective processes. Kahneman (2011) simply distinguishes between processes that are 'fast' and 'slow.' Rubinstein (2007, 2013) distinguishes between "instinctive" and "cognitive" processes. Denes-Raj and Epstein (1994) distinguish between an 'experiential system' and a 'rational system'. Hsee and Rottenstreich (2004) posit two qualitatively different types of valuation processes – valuation by feeling and valuation by calculation[1]. Following Stanovich and West (2000), we adopt the neutral System 1/System 2 distinction in our analysis.

Concurrent with the dual process paradigm developing in the psychology literature, a plethora of dual process models have emerged recently in economics, each with similar distinctions between the types of processes involved. The relation between DPU and alternative dual system or dual selves models is discussed in §14. Despite their recent rise to theoretical prominence, two-system (dual process) theories date back to the early days of scholarly thought. The conflict between reason and passion, for instance, features prominently in Plato's *Republic* and in Adam Smith's *Theory of Moral Sentiments*.

[1]We do not employ 'calculation' to mean the agent is necessarily calculating expected utilities consciously. Rather, 'calculation' is meant in a broad sense to refer to reliance on logic and reasoning to make choices.

## 3. DUAL PROCESS UTILITY THEORY

### 3.1. *Setup and Assumptions*

Our approach in this section builds on the variant of Harsanyi's (1955) preference aggregation theorem in Keeney and Nau (2011). Keeney and Nau adapted Harsanyi's theorem for agents with subjective expected utility preferences. We adapt the setting to decisions under risk and decisions over time.

Formally, we proceed as follows: There is a finite non-empty set, T, of time periods and a finite non-empty set X of consumption sequences. We index consumption sequences by $j \in \{1,2,\dots,n\}$ and we index time periods by $t \in \{0,1,\dots,T\}$. A *consumption sequence* $x_j \coloneqq [x_{j0},\dots,x_{jT}]$, consists of dated outcomes in $\mathbb{R}^n$. A *stochastic consumption plan* is a probability distribution over *consumption sequences*. We denote a stochastic consumption plan by a function $f\colon \mathrm{X} \to [0,1]$, whose elements are indexed by consumption sequences, with $f(x_j)$ denoting the probability it assigns to consumption sequence $x_j$. Denote the set of stochastic consumption plans by Ω.

We first make assumptions about the risk and time preferences of Systems 1 and 2. We first assume that the preferences for each system are internally consistent. Viewing discounted expected utility (DEU) preferences as a model of consistent risk and time preferences, we impose this structure on the risk and time preferences for each system. We will see that imposing consistent preferences for each system can nevertheless generate a variety of decision anomalies as emergent phenomena that arise through the interactions between systems. As a further restriction, we let System 1 be more risk-averse and more delay-averse (i.e., less patient) than System 2. The assumption that System 1 is more risk-averse than System 2 is broadly consistent with a range of evidence. Frederick (2005), Burks et al., (2009), Dohmen et al., (2010), and Benjamin et al. (2013) find that individuals with higher cognitive ability are less risk-averse (in particular, closer to risk-neutrality) than individuals with low levels of cognitive ability or cognitive reflection. Deck and Jahedi (2015) review evidence suggesting that increasing cognitive load (an experimental manipulation designed to increase reliance on System 1) leads to greater small-stakes risk aversion. Kirchler et al. (2015) find that placing individuals under time pressure also produces greater risk aversion (for gains). The assumption that System 1 is less patient than System 2 is consistent with findings by Frederick (2005), Burks et al. (2009), Dohmen et al. (2010), and Benjamin et al. (2013) that decision makers with higher cognitive ability or cognitive reflection are more patient than individuals with lower levels of cognitive ability. In addition, Tsukayama and Duckworth (2010) find that decision makers are less patient for affect-rich outcomes. The following assumption is thus consistent with empirical evidence regarding the relationship between the risk and time preferences of Systems 1 and 2. Formally, let $\succsim_s$ and $\succ_s$ denote weak and strict preference, respectively, between pairs of stochastic consumption plans for system $s,\ s \in \{1,2\}$.

**Assumption 1 (Preferences[2] of Systems 1 and 2):** *System $s$, $s \in \{1,2\}$ has discounted expected utility preferences, with System 1 more risk-averse and more delay-averse than System 2. That is, there exist utility functions $u_1, u_2$, with $u_1$ more concave than $u_2$, and unique discount factors, $\delta_1, \delta_2$ with $0 < \delta_1 < \delta_2 \leq 1$, such that for all $f, g \in \Omega$,*

(1) $f \succsim_s g \Leftrightarrow V_s(f) \geq V_s(g)$, *where* $V_s(f) = \sum_t \sum_j \delta_s^t \cdot f(x_j) \cdot u_s(x_{jt})$.

All of our results continue to hold even in the more restrictive case where System 2 is risk-neutral and delay-neutral. Mukherjee (2010), Loewenstein et al. (2015) and Schneider and Coulter (2015) each argue that risk-neutrality is a plausible, and even natural, assumption for System 2. To the extent that System 2 characterizes an idealized rational agent, it appears at least plausible that it does not have a pure rate of time preference which some authors have argued to be irrational (e.g., Harrod, 1948; Traeger, 2013), and that it maximizes expected value. Formally, let $(c, p, t) \in \Omega$ denote the stochastic consumption plan $f$ defined as $f(\mathrm{x}) = p$ if x is the all zeros vector except at $x_t = c$, and $f(\mathrm{x}) = 1 - p$ if x is the all zeros vector, and $f(\mathrm{x}) = 0$ for all other x ∈ X. By 'delay neutrality' we mean $(c, p, t) \sim_2 (c, p, r)$ *for all* $t, r \in T$. Hence, in this special case,

[2] We avoid explicit preference axioms for Systems 1 and 2 to simplify the exposition and to focus on the novel parameter $\theta$ in our model which is derived from our assumptions in Proposition 1. See Traeger (2013) for an axiomatization of discounted expected utility.

System 2 discounts the future for reasons of uncertainty, given the future is often more uncertain than the present, but not for reasons of impatience.

Let $\succsim$ and $\succ$ represent, respectively, weak and strict preferences of the decision maker over stochastic consumption plans. We minimally constrain the agent's time preferences, and do not impose stationarity or even time separability so that we do not rule out common behaviors such as present bias or the magnitude effect. But note that while not ruling out these behaviors, we do not assume them either. Present bias and the magnitude effect as well as some observed interactions between risk and time preferences will emerge as general properties *implied* by our representation. Formally, our Assumptions 2 and 3 can be viewed as special cases of Assumption 2 and 3 in Harsanyi's aggregation theorem, as presented in Keeney and Nau (2011).

**Assumption 2 (Preferences of the Decision Maker):** *The decision maker has time-dependent expected utility preferences among stochastic consumption plans. That is, there exists a possibly time-dependent utility function, $u_t(x)$, such that for all $f, g \in \Omega$,*

(2) $$f \succsim g \Leftrightarrow V(f) \geq V(g), \text{ where } V(f) = \textstyle\sum_t \sum_j f(x_j) \cdot u_t(x_{jt}).$$

Our final assumption is a simple Pareto condition that relates Assumptions 1 and 2.

**Assumption 3 (Pareto Efficiency):** *If both systems weakly prefer $f$ to $g$, then $f \succsim g$, and if, in addition, one system strictly prefers $f$ to $g$ then $f \succ g$.*

Given the preceding assumptions, we have the following result:

**Proposition 1 (Dual Process Utility Theorem):** *Given Assumptions 1, 2, and 3, there exists a unique constant*[3] *$\theta \in (0,1)$, unique discount factors $\delta_1, \delta_2$ with $0 < \delta_1 < \delta_2 \leq 1$ and utility functions, $u_1$ and $u_2$, with $u_1$ more concave than $u_2$, such that for all $f, g \in \Omega$, the decision maker's preferences are given by $f \succsim g \Leftrightarrow V(f) \geq V(g)$, where*

(3) $$\begin{aligned} V(f) &= (1-\theta)V_1(f) + \theta V_2(f) \\ &= (1-\theta)\left(\textstyle\sum_t \sum_j \delta_1^t \cdot f(x_j) \cdot u_1(x_{jt})\right) + \theta\left(\textstyle\sum_t \sum_j \delta_2^t \cdot f(x_j) \cdot u_2(x_{jt})\right). \end{aligned}$$

[3] In Harsanyi's theorem, the weights on individual member utilities are positive and unique up to a common scale factor. Without loss of generality, the weights can be scaled to sum to 1 in which case $\theta \in (0,1)$ is uniquely determined. Although Proposition 1 restricts $\theta \in (0,1)$, we will continue to discuss $\theta = 0$ and $\theta = 1$ as special cases of DPU, as both are limiting cases and $\theta$ can be arbitrarily close to 0 or 1.

One could imagine many ways of aggregating preferences of Systems 1 and 2. Proposition 1 provides a formal justification for the convex combination approach. Although including the time domain and the application of Harsanyi's theorem from social choice theory to model *individual* choice behavior is new, the proof for Proposition 1 follows straightforwardly from the proof of Harsanyi's theorem and so is omitted here (see Keeney and Nau (2011) for a detailed proof).

The assumption of discounted expected utility preferences seems particularly appropriate for System 2 which may be intuitively thought to resemble the rational economic agent. However, in addition to differences in the content of risk and time preferences between systems (i.e., that the systems differ in their degrees of risk aversion and impatience), one might further propose that the two systems differ in the structure of their risk and time preferences, with System 2 having normative discounted expected utility preferences, and with System 1 having behavioral preferences based on prospect theory (PT) or rank dependent utility (RDU) theory. Supporting this, Rottenstreich and Hsee (2001) find that inverse S-shaped probability weighting (as assumed in RDU theory (Quiggin 1982) and prospect theory[4] due to Tversky and Kahneman (1992)) is more pronounced for affect-rich outcomes. Support for assuming that System 1 has prospect theory preferences also comes from Barberis et al. (2013) who use prospect theory to model "System 1 thinking" for initial reactions to changes in the value of financial assets. Reflecting on prospect theory in his book, "Thinking, Fast, and Slow," Kahneman (2011) remarks, "It's clear now that there are three cognitive features at the heart of prospect theory…They would be seen as operating characteristics of System 1" (281-282).

Recent impossibility results (Mongin and Pivato, 2015; Zuber, 2016) have demonstrated that axiomatic methods cannot be used to aggregate non-expected utility preferences. For instance, Zuber (2016) considers a general class of non-expected utility preferences and concludes, "non-expected utility preferences cannot be aggregated

[4]Although this version is commonly called cumulative prospect theory, Peter Wakker has noted (personal communication, July 2, 2016) that it was Amos Tversky's preference for this latter version to be called prospect theory. Following Tversky's preference, we refer to the 1992 version as prospect theory.

consistently." However, formally we can generalize (1) by allowing for the possibility that System 1 engages in non-linear probability weighting, as in RDU theory and prospect theory. While we employ RDU theory, the examples and behaviors at the focus of our analysis involve only positive outcomes, in which case PT coincides with RDU.

In our analysis, we allow for the more general case where System 1 has behavioral preferences (given by prospect theory or rank–dependent utility theory) and System 2 has normative risk preferences given by discounted expected utility theory. Formally, we consider the following generalization of (3):

(4) $$V(f) = (1-\theta)\left(\sum_t \sum_j \delta_1^t \cdot \pi\big(f(x_j)\big) \cdot u_1\big(x_{jt}\big)\right) + \theta \sum_t \sum_j \delta_2^t \cdot f(x_j) \cdot u_2(x_{jt}).$$

where System 1 distorts cumulative probabilities by the function $\pi: [0,1] \rightarrow [0,1]$, with $\pi(0) = 0$ and $\pi(1) = 1$, that takes the standard rank-dependent form given below with weighting function $w$:

$$\pi(f(x_j)) = w\big(f(x_j) + \cdots + f(x_1)\big) - w\big(f(x_{j-1}) + \cdots + f(x_1)\big),$$

for $j \in \{1,2, \dots, n\}$, where consumption sequences are ranked according to the discounted utility for System 1 for each sequence such that $\sum_t \delta_1^t \cdot u_1(x_{nt}) \leq \cdots \leq \sum_t \delta_1^t \cdot u_1(x_{1t})$.

The rank-dependent form for $\pi$ avoids violations of stochastic dominance, and the typical inverse S-shaped form often assumed for w reflects the psychophysics of probability perception as it exhibits diminishing sensitivity from the endpoints of the probability scale. This approach for ranking consumption sequences essentially collapses each sequence into its discounted utility and then rank-dependent probability weighting is applied to these discounted utilities for System 1. This effectively reduces the outcomes to a single dimension to which RDU can be applied as it would normally be applied to lotteries with static outcomes.

We say a decision maker with preferences given by (4) has *dual process utility* (DPU) preferences. Proposition 1 brings us most of the way to (4), but does not account for non-linear probability weighting. Many of our results also hold under specification (3). However, we will employ a version of (4) in our analysis to allow for the plausible case in which System 1 has prospect theory or RDU preferences. To simplify notation and to illustrate the model with the delay-neutrality and risk-neutrality assumptions for System 2, we will employ the specification in (5) in the analysis to follow, where we drop the subscripts on System 1's discount factor and utility function:

(5) $$V(f) = (1-\theta)\left(\sum_t \sum_j \delta^t \cdot \pi\big(f(x_j)\big) \cdot u\big(x_{jt}\big)\right) + \theta \sum_t \sum_j f(x_j) \cdot x_{jt}.$$

All of our subsequent results are robust to the risk-neutrality and delay-neutrality assumptions for System 2 and continue to hold if $u_2$ is any monotonically increasing concave function with less curvature than $u_1$ and if System 2 is more patient than System 1. Our subsequent analysis demonstrates that our results hold even if the representation of System 2 preferences is parameter-free as in (5). In a sense, there is a duality between the preferences of System 1 and System 2 as represented in (5): System 1 discounts the future, but System 2 does not. System 1 weights probabilities non-linearly, but System 2 does not. System 1 has a non-linear utility function, but System 2 does not.

For decisions involving only risk, (5) reduces to a variant of the dual system model (DSM) of Mukherjee (2010). The DPU model in (5) modifies the DSM by employing a rank-dependent probability weighting function for System 1, and extends the model to encompass both risk and time preferences. Rank-dependent weighting for System 1 eliminates two undesirable features of the DSM: (i) the affective system in the DSM weights all outcomes in a lottery's support equally (which implies the affective system assigns the same value to a lottery offering a 0.01 probability of $1 million and $0 otherwise as to a lottery offering a 0.99 probability of $1 million and $0 otherwise); and (ii) the DSM violates first order stochastic dominance. Property (i) seems questionable even for System 1 as it seems plausible that there is a more positive affective response for a 0.99 probability of winning $1 million. Property (ii) is often taken as an essential requirement of adequate normative and descriptive decision theories.

As a final point regarding (5), we note that DPU can be formulated in either discrete or continuous time with discrete or continuous distributions. The value of a stochastic consumption plan, $g$, under DPU is given by (6) for the analog to (5) with continuous time and continuous distributions. In (6), let weighting function $w$ be smooth with derivative $w'$, and let $G$ denote the cumulative distribution function of $g$:

$$(6) \qquad \tilde{V}(g) \coloneqq \int_0^\infty \left( \theta \int_{-\infty}^\infty x g(x) dx + (1-\theta) \int_{-\infty}^\infty \delta^t u(x) w'(G(x)) g(x) dx \right) dt.$$

### 3.2. *Interpretation of* $\theta$

The parameter $\theta$ may be interpreted as the *degree* to which an agent is 'hard-wired' to rely on System 2. Intuitive agents then have low values of $\theta$, whereas more reflective agents have higher values of $\theta$. Alternatively, the choice may be determined by one system or the other probabilistically. Then $\theta$ may be interpreted as the *probability* the agent relies on System 2 in a given decision. A third interpretation of (4) is that $1-\theta$ is a perturbation such that the 'rational' (System 2) model is perturbed in the direction of the behavioral (System 1) model. Our results in Propositions 3 – 7 suggest even arbitrarily small perturbations from the rational model can explain the interaction effects in Table 1.

We will refer to $\theta$ as the decision maker's "cognitive type," with an agent's cognitive type becoming progressively less based on feeling and intuition and more reliant on logic and calculation as $\theta$ increases. The cases $\theta = 0$ and $\theta = 1$ can be viewed as caricatures of an agent who relies only on feeling or only on calculation when making decisions.

### 3.3. *Interpretation of* $V_1$ *and* $V_2$

There is a debate as to how one should interpret the outcomes of a decision under expected utility theory. This issue is discussed by Keeney and Nau (2011) who remark: *"There is a subtlety concerning the interpretation of the consequences… Strictly speaking, a consequence is a personal experience of an individual decision maker… A different interpretation is that a consequence is "whatever happens" when a given real alternative is chosen and a given real event occurs, regardless of how it may be perceived."(p. 10).* One plausible interpretation of $V_1$ and $V_2$ is that $V_1$ represents the *subjective value* of the outcome (the "personal experience of an individual decision maker"), and $V_2$ represents the *objective value* of the outcome ("whatever happens when a given real alternative is chosen and a given real event occurs, regardless of how it may be perceived"). The function $V$ is then a weighted average of the subjective and objective values. This interpretation unifies the two perspectives noted by Keeney and Nau.

### 3.4. *Basic Properties of DPU*

Consider two stochastic consumption plans $f$ and $g$, where $f(x_j)$ and $g(x_j)$ are the probabilities which $f$ and $g$ assign to consumption sequence $x_j$, respectively. Since a decision maker either receives one consumption sequence or another and so cannot interchange components of any arbitrary sequences, we first seek a means of objectively ranking different consumption sequences, analogous to how one would rank individual outcomes. We can then extend the standard definition of stochastic dominance from lotteries over outcomes to lotteries over consumption sequences. In particular, we say sequence $x_j$ *dominates* sequence $x_k$ if $x_{jt} \geq x_{kt}$ for all $t \in \{0,1,\ldots,T\}$, with a strict inequality for at least one $t$. We say that consumption sequences $x_1, \ldots., x_n$ are *monotonically ordered* if $x_j$ dominates $x_{j-1}$ for all $j \in \{2,\ldots,n\}$. For any monotonically ordered consumption sequences $x_1, \ldots., x_n$, we say $f$ (first-order) *stochastically dominates* $g$ if $F(x_{jt}) \leq G(x_{jt})$ for all $j \in \{1,\ldots,n\}$, and all $t \in \{0,1,\ldots,T\}$, where $F$ and $G$ are the cumulative distribution functions for $f$ and $g$, respectively. Note that this reduces to the standard definition of stochastic dominance in an atemporal setting.

**Proposition 2:** *Let* $\succsim$ *have a DPU representation as in* (4). *Then for any fixed* $\theta \in [0,1]$, $\succsim$ *satisfies the following properties over stochastic consumption plans:*

(i) *Weak Order (*$\succsim$ *is transitive and complete)*

(ii) *Continuity*

(iii) *First order stochastic dominance.*

The proofs of properties (i) and (ii) in Proposition 2 are standard so we prove only (iii). Recall that $\pi(\cdot)$ ranks sequences such that $\sum_t \delta_1^t \cdot u_1(x_{1t}) \geq \cdots \geq \sum_t \delta_1^t \cdot u_1(x_{nt})$. Note that if consumption sequences $x_1, \ldots., x_n$ are monotonically ordered, then $u_1(x_{1t}) \geq \cdots \geq u_1(x_{nt})$ for all $t \in \{0,1,\ldots,T\}$, and for any increasing function $u_1$. Thus, $\pi(\cdot)$ preserves the monotonic ordering of the sequences. If $f$ *stochastically dominates* $g$, then $\delta^t \sum_j \pi(f(x_j))u(x_{jt}) > \delta^t \sum_j \pi(g(x_j))u(x_{jt})$, for each period $t \in \{0,1,\ldots,T\}$, which implies that $V_1(f) > V_1(g)$. Since $V_2(f) > V_2(g)$, the convex combination of $V_1$ and $V_2$ ranks $f$ higher than $g$ for all $\theta \in [0,1]$.

## 4. Empirical Violations of Discounted Utility Theory

In this and the following sections, all propositions assume the decision maker has dual process utility preferences (given by (5)). Proofs of all Propositions in §4 and §5 are given in the appendix. Notably, each of these results (Propositions 3, 4, 5, 6, and 7) do not hold when $\theta = 0$ or $\theta = 1$, indicating the need for a dual process paradigm in our setup. First, we show that DPU resolves two of the most important empirical violations of discounted utility theory - present bias and the magnitude effect.

### 4.1. *Present Bias*

Systematic empirical violations of the stationarity axiom of discounted utility theory (Koopmans, 1960), such as present bias, have been well-documented in experiments (Frederick et al., 2002), and are thought to reveal time-inconsistent preferences (Laibson 1997; O'Donoghue and Rabin 1999). Formal accounts of present bias and hyperbolic discounting have often directly assumed such behavior in the functional form of the agent's preferences (e.g., Loewenstein and Prelec 1992; Laibson 1997). Surprisingly, present bias emerges as a general property of DPU without any explicit assumptions regarding hyperbolic discounting or diminishing sensitivity to delays. In fact, present bias is predicted by DPU even though each system has time consistent preferences.

As before, let $(c, p, t)$ denote a stochastic consumption plan which has one non-zero outcome $c$, to be received with probability $p$ at time $t$. We have the following definition:

**Definition 1 (Present Bias):** *Present bias* holds if for $y \in (0, c)$, and $t, \Delta > 0$, $(y, p, 0) \sim (c, p, \Delta) \implies (y, p, t) \prec (c, p, t + \Delta)$

**Proposition 3:** *Under DPU, present bias holds if and only if* $\theta \in (0,1)$.

Proposition 3 implies that DPU explains the example of the common difference effect in Table 1 demonstrated by Keren and Roelofsma (1995). In particular, a decision maker indifferent between 100 Dutch guilders for sure now and 110 Dutch guilders for sure in 4 weeks will strictly prefer 110 Dutch guilders for sure in 30 weeks over 100 Dutch guilders for sure in 26 weeks. It is also clear from the proof of Proposition 3 that present bias does not hold if $\theta = 0$ or if $\theta = 1$. Thus, under DPU, present bias arises due to the interaction between System 1 and System 2.

### 4.2. *The Magnitude Effect*

The DPU model also offers an explanation of the magnitude effect in intertemporal choice. The magnitude effect is the robust observation that behavior is more patient for larger rewards than for smaller rewards (Prelec and Loewenstein, 1991). For instance, the magnitude effect implies that a decision maker indifferent between \$1 now and \$2 in one year, will prefer \$200 in one year over \$100 now. Formally, we have:

**Definition 2 (Magnitude Effect):** We say the *magnitude effect* holds if for $y \in (0, c)$, $s > t$, and $r > 1$, $(y, p, t) \sim (c, p, s) \implies (ry, p, t) \prec (rc, p, s)$

**Proposition 4:** *For any concave power utility function $u$, the magnitude effect holds under DPU, if and only if $\theta \in (0,1)$.*

Note that a power utility function is sufficient, but not necessary for DPU to generate the magnitude effect. A power utility function is also the most widely used specification for prospect theory (Wakker and Zank, 2002) and among the most-widely used utility functions in analysis and applications of expected utility theory.

## 5. Interactions between Risk and Time Preference

In this section, we apply DPU to systematic interactions between risk and time preferences from Table 1, identified in Baucells and Heukamp (2012).

### 5.1. *Time Interacts with Risk Preference*

We return now to the behaviors illustrated in Table 1. As displayed in the table, Baucells and Heukamp (2010) found that most respondents in their study preferred a guaranteed 9 Euros immediately over an 80% chance of 12 Euros immediately, but chose the possibility of receiving 12 Euros immediately when the probabilities of winning were scaled down by a factor of 10. This behavior is an instance of the Allais common ratio effect (Allais, 1953). Baucells and Heukamp further observed that when the receipt of payment is delayed 3 months, most respondents preferred an 80% chance of 12 Euros over a guaranteed 9 Euros. This finding that people are less risk-averse toward delayed lotteries was also observed by Abdellaoui et al. (2011).

The common ratio effect example from Baucells and Heukamp (2010) holds under DPU if the probability weighting function is sub-proportional. Here we confirm that DPU explains the finding that 'time interacts with risk preference' which holds even if System 1's utility function is linear in probabilities. Let $\mathbb{E}[f]$ denote the (undiscounted) expected value of stochastic consumption plan $f$. We consider the case where the riskier lottery has the higher expectation as was the case in Baucells and Heukamp (2010).

**Definition 3:** We say *time interacts with risk preference* if for $y \in (0, c)$, $\alpha \in (0,1)$, and $s > t$ , $(y, p, t) \sim (c, \alpha p, t) \Longrightarrow (y, p, s) \prec (c, \alpha p, s)$.

**Proposition 5:** *Let* $\mathbb{E}[(c, \alpha p, t)] > \mathbb{E}[(y, p, t)]$. *Then under DPU, time interacts with risk preference if and only if* $\theta \in (0,1)$.

Interestingly, DPU accounts for changes in risk preference over time without assuming time-varying risk-preference parameters.

### 5.2. *Risk Interacts with Time Preference*

As displayed in Table 1, Keren and Roelofsma (1995) found that most respondents in their study preferred a guaranteed 100 Dutch guilders immediately over a guaranteed 110 Dutch guilders in 4 weeks, but chose the guaranteed 110 when the receipt of both payments was delayed an additional 26 weeks. This behavior is an example of present bias. Keren and Roelofsma further observed that when the chance of receiving each payment was reduced, most respondents preferred a 50% chance of 110 Dutch guilders in 4 weeks over a 50% chance of 100 now. That is, making both options risky leads to more patient behavior, analogous to the effect of adding a constant delay to both options.

**Definition 4:** We say *risk interacts with time preference* if for $y \in (0, c)$, $t, \Delta > 0$, and $q < p$, $(y, p, t) \sim (c, p, t + \Delta) \Longrightarrow (y, q, t) \prec (c, q, t + \Delta)$.

**Proposition 6:** *If* $w(q)/w(p) < q/p$, *then under DPU, risk interacts with time preference if and only if* $\theta \in (0,1)$.

In the example by Keren and Roelofsma in Table 1, the condition on the weighting function in Proposition 6 reduces to $w(0.5) < 0.5$, which implies $\pi(0.5) < 0.5$. This inequality is a general feature of observed probability weighting functions (Starmer, 2000) and represents a form of pessimism. Indeed this condition is implied by the assumption of pessimism in the rank-dependent utility framework (Chateauneuf et al., 2005). This condition holds for all convex probability weighting functions as well as for the familiar inverse-S-shaped weighting functions (such as those parameterized by Tversky and Kahneman (1992), Prelec (1998) and Gonzalez and Wu (1999)). The early weighting functions in the literature were selected primarily on their ability to fit experimental data and generate an inverse-S-shape. A recent weighting function by Blavatskyy (2016) provides a more intuitive basis for such weighting functions. Blavatskyy (2016) formally demonstrates that for the inverse-S-shaped weighting function he proposes, the condition $\pi(0.5) < 0.5$ implies "an aversion to the dispersion of outcomes and an attraction to positively skewed distributions." (p. 103). The condition $\pi(0.5) < 0.5$ will reappear in our analysis and is the only substantive property of the weighting function that is necessary for DPU to explain the experimental observations studied here. The more general condition $w(q)/w(p) < q/p$ is only necessary for the generalization of the behavior observed by Keren and Roelofsma to all $q < p$ as formalized in Definition 4. This latter condition is satisfied, for instance, by any convex power weighting function.

### 5.3. *Payoffs Interact with Risk and Time Preferences*

Baucells et al. (2009) find that 81% of respondents preferred €100 for sure in one month to €100 with 90% immediately, but 57% preferred €5, with 90% immediately over €5 for sure, in one month. Baucells and Heukamp (2012) refer to this behavior as subendurance and they define it more generally as follows:

**Definition 5:** *Subendurance* holds if for $y \in (0, c)$, $t, \Delta > 0$ and $\lambda \in (0,1)$, $(c, p, t + \Delta) \sim (c, \lambda p, t) \Longrightarrow (y, p, t + \Delta) \prec (y, \lambda p, t)$.

**Proposition 7:** *For any concave power utility function $u$, subendurance holds*[5] *under DPU, if and only if $\theta \in (0,1)$.*

Note that the interaction effects discussed in this section (effect of time on risk preference, effect of risk on time preference, subendurance) challenge a larger class of

[5] The proof of Proposition 7 shows that subendurance holds more generally if $u(c)/u(y) < c/y$ for $y < c$.

time preferences than discounted utility theory. Indeed, they cannot be explained by any model of discounting in which the evaluation of payoffs, probabilities, or delays is multiplicatively separable. Yet the major models in the literature typically assume that time preference is separable into a discount function and a utility function. The class of such separable time preferences includes traditional discounted utility theory as well as the model of hyperbolic discounting by Loewenstein and Prelec (1992), the model of quasi-hyperbolic discounting studied in Laibson (1997), and more recent models involving both risk and delay such as those by Halevy (2008) and Epper and Fehr-Duda (2015). As, Baucells and Heukamp (2012) note, when evaluating a stochastic consumption plan $(x, p, t)$, "One may be tempted to propose $V(x, p, t) = w(p)f(t)v(x)$. Unfortunately, this form is not appropriate because by Patterns 1 – 3 and 4 – 6, probability and time cannot be separated. One may then propose the more general form $V(x, p, t) = g(p, t)v(x)$, but this fails to accommodate subendurance." [6] Moreover, Ericson and Noor (2015) reject the common assumption that discount functions and utility functions are separable for nearly 70% of their participants. Given the necessity of a seemingly complex non-separable functional form for evaluating $(x, p, t)$ in order to explain the observations in Table 1, the DPU functional form in (5) is surprisingly simple. Under an even simpler special case of (5) in which System 1 has expected utility preferences, Propositions 2, 3, 4, 5, and 7 continue to hold. Note also that $\theta \in (0,1)$ is necessary to explain present bias, the magnitude effect, and interaction effects between time and risk preferences in Table 1. We show in §9 that $\theta$ also has implications for correlations between risk preference, time preference, and cognitive ability.

## 6. Risk Preference and Intertemporal Substitution

The discounted expected utility model uses the same utility function in both risky and temporal contexts. However, risk preference and inter-temporal substitution are often observed to be distinct (e.g., Miao and Zhong, 2015). Consider the following stochastic

[6] In their quotation, Pattern 1 – 3 refers to the observation that delay reduces risk aversion and Pattern 4-6 refers to the finding that risk reduces impatience.

consumption plan, $f$, also considered by Miao and Zhong (2015), subject to $(1+r)c_1 + c_2 = 100$ and $(1+r)c_1' + c_2' = 100$, where $r \in (0,1)$ is an interest rate.

| | $t_1$ | $t_2$ |
|---|---|---|
| $p$ | $c_1$ | $c_2$ |
| $1-p$ | $c_1'$ | $c_2'$ |

The present equivalents $\mathrm{PE}(c_1, c_2)$ and $\mathrm{PE}'(c_1', c_2')$ of consumption $(c_1, c_2)$ and $(c_1', c_2')$, respectively, are determined such that PE/PE$'$ at $t_1$ is indifferent under V to receiving $(c_1, c_2)/(c_1', c_2')$ on the time horizon. They are defined as

$$PE(c_1, c_2) := V^{-1}\big(\,(1-\theta)(u(c_1) + \delta u(c_2)) + \theta(c_1 + c_2)\big).$$

$$PE'(c_1', c_2') := V^{-1}\big((1-\theta)(u(c'_1) + \delta u(c'_2)) + \theta(c'_1 + c'_2)\big).$$

Employing rank-dependent probability weighting to aggregate the certainty equivalent as in the Chew-Epstein-Halevy approach (see Miao and Zhong, 2015), the certainty equivalent (CE) under DPU can be expressed as

$$CE(f) = V^{-1}\left(w(p)V\big(PE(c_1, c_2)\big) + w(1-p)V\big(PE(c_1', c_2')\big)\right) if\ PE \geq PE'$$

$$CE(f) = V^{-1}\left(w(1-p)V\big(PE(c_1, c_2)\big) + w(p)V\big(PE(c_1', c_2')\big)\right) if\ PE \leq PE'.$$

This approach permits a separation between risk attitude (which is partially determined by $w$) and inter-temporal substitution (which does not depend on $w$).

## 7. Preference for Diversification Across Time

Miao and Zhong (2015) provide a variant of the following example in which they observe a preference for diversification across time:

| Option A | $t=0$ | $t=1$ |
|---|---|---|
| $p = 0.5$ | 100 | 0 |
| $1-p = 0.5$ | 0 | 100 |

| Option B | $t=0$ | $t=1$ |
|---|---|---|
| $p = 0.5$ | 100 | 100 |
| $1-p = 0.5$ | 0 | 0 |

We can think of the consumption sequences as being determined by the toss of a fair coin. Then Option A offers \$100 in period 0 (and nothing in period 1) if the coin lands heads, and it offers \$100 in period 1 (and nothing in period 0) if the coin lands tails. In contrast, Option B offers \$100 in period 0 and \$100 in period 1 if the coin lands heads, and it pays \$0 in both periods if the coin lands tails. Miao and Zhong (2015) propose and find experimental support for the hypothesis that in such choices, a decision maker would prefer Option A in which risks are diversified across time over Option B in which they are not. Such behavior has also been observed by Andersen et al. (2011) who refer to this preference pattern as 'correlation aversion' or 'intertemporal risk aversion.'

Correlation aversion is simply explained by DPU. Note that, for Option A, System 1 will rank consumption sequence $x \coloneqq (100, t = 0; 0, t = 1)$ higher than the sequence $y \coloneqq (0, t = 0; 100, t = 1)$ in order of preference due to impatience (since $\delta < 1$). Thus, DPU assigns weight $\pi(0.5)$ to sequence $x$ and weight $(1 - \pi(0.5))$ to sequence $y$. Weights are assigned analogously for Option B.

In most experimental studies of rank-dependent probability weighting functions including Tversky and Kahneman (1992), Wu and Gonzalez (1996), Gonzalez and Wu (1999), Abdellaoui (2000), and Bleichrodt and Pinto (2000), it has been found that for a binary lottery $(x, p; y, 1 - p)$, where $x > y$, when $p = 0.5$, the rank-dependent weight is given by $\pi(0.5) < 0.5$ (see also Starmer (2000) and Wakker (2010)). This condition is also a general property resulting from Prelec's (1998) axiomatic characterization of the probability weighting function. Under DPU, with $u(0) = 0$, the values of the options are:

$$V(Option\ A) = (1 - \theta)\left(\left(\pi(0.5)\right)u(100) + \left(1 - \pi(0.5)\right)\delta u(100)\right) + \theta(100)$$

$$V(Option\ B) = (1 - \theta)\left(\left(\pi(0.5)\right)\left(u(100) + \delta u(100)\right)\right) + \theta(100)$$

Since $\delta \in (0,1)$, $u(100) > \delta u(100)$. Hence, A is preferred to B if $\pi(0.5) < 0.5$ which is a robust finding. Thus, DPU predicts a preference for diversification across time.

## 8. AVERSION TO TIMING RISK

Onay et al. (2007) and DeJarnette et al. (2015) experimentally investigate preferences over lotteries that pay a fixed prize at an uncertain date. For instance, in choices such as receiving \$100 in 10 weeks for sure (Option A), or receiving \$100 in either 5 or 15 weeks with equal probability (Option B), they find that people are generally risk-averse toward timing risk, preferring Option A. However, DEU and the standard models of hyperbolic and quasi-hyperbolic discounting imply people will be *risk-seeking* toward timing risk.

As in DeJarnette et al., consider a choice between receiving \$100 at time $t$ (Option A), or \$100 at either time $t-r$ or time $t+r$ with equal probability (Option B). Under DPU, the values of Options A and B are given by:

$V(A) = (1-\theta)\delta^{t}u(100) + \theta(100).$

$V(B) = (1-\theta)\left(\delta^{t-r}\pi(0.5)u(100) + \delta^{t+r}\big(1-\pi(0.5)\big)u(100)\right) + \theta(100).$

For all $\theta \in [0,1)$, Option A is preferred to Option B if the following inequality holds:

(7) $$1 > [\delta^{-r}\pi(0.5) + \delta^{r}\big(1-\pi(0.5)\big)].$$

The above inequality can hold given $\pi(0.5) < 0.5$, generating a preference for receiving payment in 10 weeks. The condition $\pi(0.5) < 0.5$ is a robust finding, noted in §5.2.

## 9. Risk Preference, Time Preference, and Cognitive Ability

The DPU model also captures observed relationships between risk preference, time preference, and cognitive ability. An agent's 'cognitive type', as parameterized by $\theta$ can be interpreted as a measure of reliance on System 2 processing which may be correlated with cognitive ability. DPU accommodates a continuum of types - any $\theta \in [0,1]$. Note that the DPU specifications in (5) predicts the following properties of $\theta$:

**Proposition 8:**

(i) *The decision maker approaches risk-neutrality as $\theta$ increases.*

(ii) *The decision maker becomes more patient as $\theta$ increases.*

(iii) *Expected value maximization is negatively correlated with impatience.*

To the extent that $\theta$ indexes an agent's cognitive ability, DPU predicts the correlations between risk neutrality and cognitive ability observed by Frederick (2005), Burks et al. (2009), Oechssler et al. (2009), Cokely and Kelley (2009), Dohmen et al. (2010), and Benjamin et al. (2013). Burks et al. (2009) report "those individuals making choices just shy of risk-neutrality have significantly higher CS [cognitive skills] than those making more either risk-averse or more risk-seeking choices" (p. 7747). However, Andersson et al. (2016) finds no correlation between risk preferences and cognitive abilities.

The DPU model also predicts the correlations between time preference and cognitive ability and between risk preference and time preference that have been found in a variety of studies (e.g., Frederick (2005), Burks et al. (2009), Dohmen et al. (2010), and Benjamin et al. (2013)). These studies observed that agents with greater cognitive ability are also typically more patient than those with lower cognitive ability, and that the tendency to maximize expected value is negatively correlated with impatience.

The notion that System 2 is closer to risk-neutrality and is more patient than System 1 is also supported by studies which employ other means of manipulating System 1 versus System 2 processing. Placing people under a high working memory load for instance, is one approach to inducing greater reliance on System 1 processing. Studies have found that increased cognitive load (Deck and Jahedi, 2015; Holger et al., 2016) and time pressure (Kirchler et al., 2015) both increase deviations from risk-neutrality such as increased small-stakes risk aversion. Cognitive load has also been found to produce less patient and more impulsive behavior (Shiv and Fedorikhin, 1999). Leigh (1986), and Anderhub et al. (2001) also find risk aversion is positively correlated with impatience.

Outside the laboratory, DPU is consistent with the finding that higher cognitive ability is associated with stock market participation, even when controlling for wealth, income, age, and other demographic information (Cole and Shastry, 2008; Grinblatt et al., 2011).

## 10. Heterogeneity in Risk Taking

Bruhin et al. (2010) characterize risk taking behavior by estimating a finite mixture model for three experimental data sets. They report “Roughly 80% of the subjects exhibit significant deviations from linear probability weighting of varying strength,” and that “Twenty percent of the subjects weight probabilities near linearly and behave essentially as expected value maximizers.” (p. 1375). Similar heterogeneity was observed by Harrison and Rutstrom (2009) who also found the modal EU type to be risk-neutral. Since DPU reduces to risk-neutrality when $\theta = 1$, and it reduces to RDU when $\theta = 0$, it can capture the observed distribution of risk preferences. Rather than interpreting the mixture model as proportions of a population who are either RDU *or* EU types, DPU offers a unified perspective in which agents are a mixture of both RDU *and* EU types.

## 11. Changes in Process Across Contexts

All results in the previous sections hold even if $\theta$ is held fixed. Here we consider additional implications of DPU if $\theta$ changes systematically across contexts such that decisions requiring more contemplation or calculation induce higher values of $\theta$ and 'easy' or intuitive, or tempting decisions induce lower values of $\theta$. That is, we consider the possibility that systematic changes in process could produce systematic changes in preference. Two cases in which $\theta$ may change systematically with the decision are:

(i) $\theta$ may decrease for affect-rich outcomes, as the agent relies more on feeling.

(ii) $\theta$ may increase for pricing tasks, as the agent relies more on calculation.

### 11.1. *Temptation*

In the DPU model, probability weighting emerges from a feeling-based System 1, which governs intuitive and affective processes[7]. Under the plausible assumption that $\theta$ decreases as outcomes become more affectively appealing, DPU predicts the following:

**Prediction 1:** *Nonlinear probability weighting is more pronounced for affect-rich outcomes (e.g., candy) than for affect-poor outcomes (e.g., money).*

Prediction 1 follows since greater weight is placed on the probability weighting function as $\theta$ decreases. This prediction is supported by the experimental results of Rottenstreich and Hsee (2001).

**Prediction 2:** *Affect-rich outcomes are discounted more than affect-poor outcomes.*

Prediction 2 holds under DPU since decreasing $\theta$ for affect-rich outcomes places greater weight on the affective discount factor, and correspondingly reduces the weight on the more patient System 2 value function. This prediction is also supported by the results of Tsukayama and Duckworth (2010) who observed greater discounting for 'tempting' rewards (candy, chips, beer) than for rewards that were not seen as tempting. Prediction 2 is also consistent with the evidence that increased cognitive load makes temptation harder to resist (Shiv and Fedorikhin, 1999).

[7] The diminishing sensitivity property of perception is another determinant of the form of probability weighting functions. See Rottenstreich and Hsee (2001), section 7.1 in Wakker (2010), and references therein, for a discussion of perceptual, emotional, and cognitive influences on probability weighting.

### 11.2 *Preference Reversals under Risk and over Time*

All results in the previous sections hold even if $\theta$ is fixed. Additional implications arise if $\theta$ may change systematically across decision tasks. For instance, it appears plausible that tasks which require calculation (such as pricing tasks) might elicit more calculation-based or analytical processing. Such tasks may thus increase $\theta$, thereby increasing the weight on the System 2 value function. This hypothesis explains empirically observed preference reversals both under risk (as observed by Lichtenstein and Slovic (1971), and Tversky et al. (1990)) and over time (as observed by Tversky et al. (1990)). In preference reversals under risk, a decision maker chooses between a high probability, low payoff bet (a P-bet) and a low probability, high payoff bet (a $-bet) which usually has a higher expected value[8]. The decision maker subsequently prices each lottery in isolation. The robust finding observed by Lichtenstein and Slovic (1971), Grether and Plott (1979), Tversky et al. (1990) and others is that the P-bet is preferred in a choice task, but the $-bet is assigned a higher minimum selling price than the P-bet.

Under the assumption that pricing tasks induce more calculations than choice tasks (or more concretely, that the weight on $V_2$ is higher in pricing than in choice tasks), DPU predicts that preferences will shift toward expected value maximization in pricing tasks. Thus, a DPU agent who is indifferent between the P-bet and the $-bet in a choice task, will assign a higher value to the $-bet in a pricing task, as observed.

Note that under the same assumption that calculation-based processing is higher in pricing tasks than choice tasks, DPU makes the novel prediction that pricing tasks will induce more patient behavior. This prediction is supported by the time preference reversals observed by Tversky et al. (1990) who found that a smaller sooner payoff was chosen 74 percent of the time, but it was priced higher than the larger delayed payment only 25 percent of the time. Under DPU, if there is a higher weight on $V_2$ in pricing than choice, a decision maker who is indifferent between the smaller sooner and larger delayed payoffs in the choice task will assign a higher price to the larger delayed payoff.

[8] In the study by Tversky et al. (1990), for 16 of the 18 lottery pairs, the $-bet had a higher expected value.

## 12. SUMMARY OF RESULTS

A summary of behaviors explained by DPU is provided in Table 2. For the behaviors from present bias through subendurance, Table 2 displays the necessary restrictions on (3) to generate the effects. For the remaining behaviors, Table 2 displays the necessary restrictions on (4) to generate the effects. As noted in §5.2, the condition $w(0.5) < 0.5$ is a standard finding in the empirical literature which holds for typical estimates of the characteristic inverse-S shape as well as for any convex weighting function. Even with these relatively standard and innocuous restrictions, the model predicts a diverse set of behaviors in the direction observed in experimental studies. In addition, all properties hold while preserving transitivity, continuity, and stochastic dominance.

**Table 2. Necessary Conditions for DPU to Explain Observed Behaviors**

| **Property** | **Parameter Values*** |
|---|---|
| **Present bias** (Laibson, 1997) | $\theta \in (0,1)$, $\delta_1 < \delta_2$ |
| **Delay reduces risk aversion** (Baucells and Heukamp, 2010) | $\theta \in (0,1)$, $\delta_1 < \delta_2$ |
| **Cognitive type and time preference** (Burks et al., 2009) | $\delta_1 < \delta_2$ |
| **Cognitive type and risk preference** (Burks et al., 2009) | $u_1$ more concave than $u_2$ |
| **Patience correlates with risk neutrality** (Burks et al., 2009) | $u_1$ more concave than $u_2$, $\delta_1 < \delta_2$ |
| **Magnitude effect** (Loewenstein & Prelec, 1991) | $\theta \in (0,1), u_1$ more concave than $u_2$ |
| **Subendurance** (Baucells & Heukamp, 2012) | $\theta \in (0,1), u_1$ more concave than $u_2$ |
| **Risk reduces impatience** (Keren & Roelofsma, 1995) | $\theta \in (0,1), w(0.5) < 0.5$ |
| **Diversifying risks across time** (Miao & Zhong, 2015) | $\theta \in [0,1)$, $w(0.5) < 0.5$ |
| **Aversion to timing risk** (DeJarnette et al., 2015) | $\theta \in [0,1), w(0.5) < 0.5$ |
| **Separation of risk and time preference** (Epstein & Zin, 1989) | $\theta \in [0,1), w(p) \neq p$ |

* The parameter $\theta$ is restricted such that $\theta \in [0,1]$ unless otherwise specified.

## 13. Parameter Elicitation

Here we briefly note a strategy for eliciting the parameters of the DPU specification in (5) directly from qualitative preference data: First, consider lotteries with no delays and with the same expected values. In this case, (5) reduces to RDU and standard techniques for eliciting the RDU parameters (functions $u$ and $w$) can be used. For instance, one may use the two-step parameter-free elicitation technique proposed by Abdellaoui (2000). After eliciting the RDU functions, present subjects with simple prospects of the form $(x, p, t)$ (with $x > 0$ to be received with probability $p$ at time $t$, and 0 otherwise) that have different delays but with the same undiscounted expected value to identify $\delta$. In particular, holding the expectation fixed, the value of prospect $(x, p, t)$ under (5) reduces to $\delta^t w(p)u(x)$. Given an immediate prospect, $(x, p, 0)$, and a prospect delayed one time period $(y, q, 1)$, one can vary $x, p, y$, and $q$ (subject to the constraint $xp = yq$) using survey data to elicit an indifference point such that $(x, p, 0) \sim (y, q, 1)$. Since the $u$ and $w$ functions have already been elicited, we obtain $\delta = w(p)u(x)/w(q)u(y)$. To elicit $\theta$, compare simple prospects with different expected values. Under this case, the value of a prospect $(x, p, t)$ under (5) is given by $(1-\theta)\delta^t w(p)u(x) + \theta px$. Given an indifference between two prospects $(x, p, t) \sim (y, q, r)$ with $xp \neq yq$, there is a unique value of $\theta$ such that $(1-\theta)\delta^t w(p)u(x) + \theta px = (1-\theta)\delta^r w(q)u(y) + \theta qy$. In particular:

$$\theta = \frac{[\delta^t w(p)u(x) - \delta^r w(q)u(y)]}{[\delta^t w(p)u(x) - \delta^r w(q)u(y) + qy - px]}.$$

This completes the approach for eliciting the DPU parameters from qualitative preference data. Note that a novel, testable prediction of DPU is that one should elicit the same value of $\theta$ from risk preference data as from time preference data.

A subtlety in the approach described here is that the System 1 and System 2 utility functions are each unique up to a positive affine transformation, although they are uniquely defined upon normalization to a 0-1 scale. Thus, both System 1 and System 2 utilities can be normalized to a 0-1 scale to facilitate comparison between System 1 and System 2 preferences.

## 14. Related Literature

A large array of models for decisions under risk and decisions over time has been developed in the past five decades, and it is not feasible to review them all here. Since models developed for only decisions under risk or for only decisions over time cannot account for the majority of our results, we focus on models which consider both risk and time. The standard discounted expected utility model motivated our analysis and it provides a natural benchmark with which to compare our predictions. Table 2 may be viewed as a summary of properties of DPU that are not shared by DEU. Prelec and Loewenstein (1991) noted parallels between anomalies for decisions under risk and decisions over time, and Loewenstein and Prelec (1992) provided a general model which accounts for observed violations of DEU such as hyperbolic discounting and the Allais paradox. Rubinstein (1988, 2003) and Leland (1994, 2002) provided models of similarity judgments which explain many of the key anomalies for decisions under risk and over time such as the Allais paradox and hyperbolic discounting as arising from the same cognitive process. However, all of these approaches treat risk and time independently, and thus cannot explain interaction effects between risk and time preferences, nor can they explain correlations between risk preference, time preference and cognitive ability.

Classic approaches to studying interactions between risk and time preferences can be found in Kreps and Porteus (1978), Epstein and Zin (1989), and Chew and Epstein (1990). Kreps and Porteus consider preferences for the timing of the resolution of uncertainty, an issue not studied here. Epstein and Zin (1989) and Chew and Epstein (1990) provide models which can disentangle risk preferences from the degree of intertemporal substitution. Traeger (2013) introduces a model of intertemporal risk aversion in which a rational agent does not discount the future for reasons of impatience.

Recent models by Halevy (2008), Walther (2010) and Epper and Fehr-Duda (2015) focus on implications of rank-dependent utility theory when extended to an intertemporal framework. Halevy (2008) and Walther (2010) focus primarily on relationships between hyperbolic discounting over time and non-linear probability weighting under risk. Halevy notes that his model is also consistent with the experimental evidence of Keren and

Roelofsma (1995). The observations of Keren and Roelofsma and Baucells and Heukamp (2010) are both explained by the probability-time tradeoff model of Baucells and Heukamp (2012). However, this model applies only to a restrictive class of prospects offering a single non-zero outcome to be received with probability $p$ at time $t$. The probability weighting model of Epper and Fehr-Duda (2015) extends the approach of Halevy (2008) by further exploring implications of RDU for decisions over time. Epper and Fehr-Duda assume that delayed prospects are transformed to include a zero outcome, reflecting the possibility that a guaranteed future payment will not materialize. They assume the likelihood of this zero outcome is determined by a constant per-period stopping probability which is transformed by a rank-dependent weighting function exhibiting subproportionality[9]. Epper and Fehr-Duda demonstrate that their model explains the interaction effects observed by Keren and Roelofsma (1995) and Baucells and Heukamp (2010) as well as preferences for the timing of the resolution of uncertainty. However, the model in Epper and Fehr-Duda (2015) does not account for subendurance or the correlations between risk and time preferences and cognitive ability.

Aside from extensions of RDU to intertemporal choice, one other major literature stream which has grown rapidly in recent years is the class of dual-selves models motivated to explain temptation and self-control as well as more general choices under risk and over time. In these models, the two families of processes have been characterized as controlled and automatic (Benhabib and Bisin, 2005), long-run and short-run (Fudenberg and Levine, 2006; 2011, 2012), hot and cold (Bernheim and Rangel, 2005), affective and deliberative (Mukherjee, 2010; Loewenstein et al., 2015), and rational and emotional (Bracha and Brown, 2012). In addition, Gul and Pesendorfer, (2001; 2004) model agents who have temptation preferences and commitment preferences.

A leading example in the class of dual-selves models is that of Fudenberg and Levine (2006, 2011, 2012) and Fudenberg et al. (2014) which can explain the Allais paradox as well as the interactions between risk and time preferences identified by Keren and

[9] The model by Epper and Fehr-Duda (2015) is specified only for the case of two non-zero outcomes, although they note that their approach can be easily generalized to $n > 2$ outcomes, provided that survival risk does not change the rank-order of the prospects.

Roelofsma and Baucells and Heukamp. However, Fudenberg et al. (2014) comment "Unfortunately the model of Fudenberg and Levine (2011) is fairly complex, which may obscure some of the key insights and make it difficult for others to apply the model." (p. 56). In addition, a major drawback of the model from both a normative and a descriptive viewpoint is that it violates transitivity (Fudenberg et al., 2014), even though transitivity is rarely violated in experiments (Baillon et al., 2014; Regenwetter et al., 2011).

Aside from the work of Fudenberg and Levine, most dual-selves models in economics are restricted to either risk or time. As noted earlier, Mukherjee's (2010) model is restricted to decisions under risk and can violate stochastic dominance. McClure et al. (2007) employ a two-system model of time preference with two discount factors but with a single utility function. Their approach can explain present bias, but not the magnitude effect or any of the interaction effects involving risk and time discussed here. The dual selves model of Bracha and Brown (2012) applies to risk and uncertainty. The models of Gul and Pesendorfer (2001; 2004), Benhabib and Bisin (2005), and Bernheim and Rangel (2005) were developed for decisions over time. The model of Loewenstein et al. (2015) is applied separately to risk and time.

Although not from a dual system perspective, mixture models of prospect theory and expected utility have been applied to experimental data at the aggregate level to identify subjects as either 'prospect theory types' *or* 'expected utility types' (Harrison and Rutstrom 2009; Bruhin et al., 2010). These empirical models account for a large degree of heterogeneity in behavior. The DPU model offers an alternative interpretation in which the decision maker is a mixture of prospect theory types *and* expected utility types.

Our results also relate to the finding in the social choice literature that present bias emerges from aggregating individual discount factors (Jackson and Yariv, 2015) in a setting of certainty where all members of society have identical utility functions. We show a similar phenomenon in a dual system model of individual choice. Conversely, our results have implications for the simultaneous aggregation of risk and time preferences in group decision making – that such preferences will display the magnitude effect and some systematic interaction effects between risk, time, and money from Table 1.

## 15. CONCLUSION

Economic models of choice are great simplifications of the complexity of actual decisions. For instance, everything that affects a person's time preferences is summarized in a single parameter, the discount factor, and everything that affects a decision maker's preferences over lotteries or consumption plans is summarized in her utility function.

In the present paper, an additional parameter has been given to the decision maker: Everything that affects a person's thinking style (e.g. whether the decision maker relies more on feeling and intuition or reason and calculation) is summarized by the parameter $\theta$ representing one's 'cognitive type', which is derived from our assumptions in Proposition 1. We saw that any $\theta \in (0,1)$ explains (i) present bias and the magnitude effect, (ii) interaction effects such as how risk affects time preference and how time affects risk preference, and (iii) observed correlations between risk preference, time preference, and cognitive ability. These observations hold under general conditions in Propositions 3 – 8 (for any $\theta \in (0,1), \delta \in (0,1)$, and without relying on a form for $w$), and they do not hold in the special cases of rank dependent utility or expected utility preferences (i.e., where $\theta = 0$ or $\theta = 1$), motivating the need for a more general theory. The DPU model also makes strong predictions as it rules out the opposite preference patterns to those in Propositions 3 – 8. In addition, DPU generates a separation between risk preference and time preference, a preference for diversification across time, and risk aversion toward timing risk. Moreover, these observations are explained under DPU, while preserving transitivity, continuity, and first order stochastic dominance.

In addition to providing a unified approach to modeling risk preference and time preference, DPU also provides a unification of three broad classes of decision models – models based on a rational economic agent (corresponding to $\theta = 1$), models based on prospect theory or rank-dependent utility theory (corresponding to $\theta = 0$), and dual system or dual selves models of decision making (corresponding to any $\theta \in (0,1)$).

## APPENDIX: PROOFS OF PROPOSITIONS

In the following proofs for Propositions 3 – 7, the agent is assumed to have DPU preferences given by (5).

**Proposition 3:** Present bias holds if and only if $\theta \in (0,1)$.

**Proof:** (Sufficiency) We need to show that (8) implies (9):

(8) $V(y,p,0) = (1-\theta)w(p)u(y) + \theta py = V(c,p,\Delta) = (1-\theta)\delta^{\Delta}w(p)u(c) + \theta pc$

(9) $(1-\theta)\delta^{t}w(p)u(y) + \theta py < (1-\theta)\delta^{t+\Delta}w(p)u(c) + \theta pc$

Note that since $c > y$, equation (8) implies that $w(p)u(y) > \delta^{\Delta}w(p)u(c)$.

Also note that (8) can be rewritten as:

$$(1-\theta)\left(w(p)u(y) - \delta^{\Delta}w(p)u(c)\right) = \theta p(c-y)$$

In addition, the inequality in (9) can be rewritten as:

$$(1-\theta)\delta^{t}\left(w(p)u(y) - \delta^{\Delta}w(p)u(c)\right) < \theta p(c-y)$$

Thus, $(1-\theta)\delta^{t}\left(w(p)u(y) - \delta^{\Delta}w(p)u(c)\right) < (1-\theta)\left(w(p)u(y) - \delta^{\Delta}w(p)u(c)\right)$.

The above inequality holds since $w(p)u(y) > \delta^{\Delta}w(p)u(c)$.

(Necessity) Under DPU, the agent has a constant discount factor if $\theta = 0$ or $\theta = 1$. ■

**Proposition 4:** For any concave power function $u$, the magnitude effect holds if and only if $\theta \in (0,1)$.

**Proof:** (Sufficiency) We need to show that (10) implies (11):

(10) $V(y,p,t) = (1-\theta)\delta^{t}w(p)u(y) + \theta py = V(c,p,s) = (1-\theta)\delta^{s}w(p)u(c) + \theta pc$

(11) $(1-\theta)\delta^{t}w(p)u(ry) + \theta pry < (1-\theta)\delta^{s}w(p)u(rc) + \theta prc$

Note that since $c > y$, equation (10) implies that $\delta^{t}w(p)u(y) > \delta^{s}w(p)u(c)$.

Also note that (10) can be rewritten as

(12) $(1-\theta)\left(\delta^{t}w(p)u(y) - \delta^{s}w(p)u(c)\right) = \theta p(c-y)$

In addition, the inequality in (11) can be rewritten as:

$$(1-\theta)\left(\delta^{t}w(p)u(ry) - \delta^{s}w(p)u(rc)\right) < \theta pr(c-y)$$

For a concave power utility function over gains, (i.e., $u(z) = z^{\alpha}$, with $z > 0, \quad \alpha < 1$), the above inequality becomes

(13) $(1-\theta)r^{\alpha}(\delta^t w(p)y^{\alpha} - \delta^s w(p)c^{\alpha}) < \theta pr(c-y)$.

Note that by (12), we have $(1-\theta)(\delta^t w(p)y^{\alpha} - \delta^s w(p)c^{\alpha})/\theta p(c-y) = 1$.

Thus, (13) reduces to $r > r^{\alpha}$, which is satisfied since $r > 1$ and $\alpha < 1$.

(Necessity) If $\theta = 0$ or $\theta = 1$, the scaling constant factors out. ■

**Proposition 5:** Let $\mathbb{E}[(c, \alpha p, t)] > \mathbb{E}[(y, p, t)]$. Then time interacts with risk preference if and only if $\theta \in (0,1)$.

**Proof:** (Sufficiency) We need to show that (14) implies (15):

(14) $(1-\theta)\delta^t w(p)u(y) + \theta py = (1-\theta)\delta^t w(\alpha p)u(c) + \theta\alpha pc$

(15) $(1-\theta)\delta^s w(p)u(y) + \theta py < (1-\theta)\delta^s w(\alpha p)u(c) + \theta\alpha pc$.

Note that since $\mathbb{E}[(c, \alpha p, t)] > \mathbb{E}[(y, p, t)]$, we have $\alpha cp > py$, in which case equation (14) implies that $\delta^t w(p)u(y) > \delta^t w(\alpha p)u(c)$. Also note that (14) can be rewritten as

$$(1-\theta)\delta^t\big(w(p)u(y) - w(\alpha p)u(c)\big) = \theta p(\alpha c - y)$$

In addition, note that the inequality in (15) can be rewritten as:

$$(1-\theta)\delta^s\big(w(p)u(y) - w(\alpha p)u(c)\big) < \theta p(\alpha c - y)$$

Thus, $(1-\theta)\delta^s\big(w(p)u(y) - w(\alpha p)u(c)\big) < (1-\theta)\delta^t\big(w(p)u(y) - w(\alpha p)u(c)\big)$.

The above inequality holds since $w(p)u(y) > w(\alpha p)u(c)$.

(Necessity) If either $\theta = 0$ or $\theta = 1$, the discount factors cancel when comparing the two alternatives and time does not affect risk preference. ■

**Proposition 6:** If $w(q)/w(p) < q/p$, risk interacts with time preference if and only if $\theta \in (0,1)$.

**Proof:** (Sufficiency) We need to show that (16) implies (17):

(16) $(1-\theta)\delta^t w(p)u(y) + \theta py = (1-\theta)\delta^{t+\Delta} w(p)u(c) + \theta pc$

(17) $(1-\theta)\delta^t w(q)u(y) + \theta qy < (1-\theta)\delta^{t+\Delta} w(q)u(c) + \theta qc$

Note that since $c > y$, equation (16) implies $\delta^t w(p)u(y) > \delta^{t+\Delta} w(p)u(c)$, and therefore $u(y) > \delta^{\Delta}u(c)$. Also note that (16) can be rewritten as:

$$(1-\theta)\delta^t w(p)\left(u(y)-\delta^\Delta u(c)\right)=\theta p(c-y)$$

In addition, note that the inequality in (17) can be rewritten as:

$$(1-\theta)\delta^t w(q)\left(u(y)-\delta^\Delta u(c)\right)<\theta q(c-y)$$

Then by (16), $\left((1-\theta)\delta^t\left(u(y)-\delta^\Delta u(c)\right)\right)/\theta(c-y)=p/w(p)$

By (17), $\left((1-\theta)\delta^t\left(u(y)-\delta^\Delta u(c)\right)\right)/\theta(c-y)<q/w(q)$.

Thus, if $w(q)/w(p)<q/p$ then (16) implies (17).

(Necessity) If either $\theta=0$ or $\theta=1$, the probability weights cancel when comparing the two alternatives and risk does not affect time preference. ■

**Proposition 7:** For any concave power function $u$, subendurance holds if and only if $\theta\in(0,1)$.

**Proof:** (Sufficiency) We need to show that (18) implies (19):

(18) $\quad (1-\theta)\delta^{t+\Delta}w(p)u(c)+\theta pc=(1-\theta)\delta^t w(\lambda p)u(c)+\theta\lambda pc$

(19) $\quad (1-\theta)\delta^{t+\Delta}w(p)u(y)+\theta py<(1-\theta)\delta^t w(\lambda p)u(y)+\theta\lambda py$

Since $pc>\lambda pc$, equation (18) implies $\delta^t w(\lambda p)u(c)>\delta^{t+\Delta}w(p)u(c)$. Also note that (18) can be rewritten as:

(20) $\quad (1-\theta)\left(\delta^t w(\lambda p)u(c)-\delta^{t+\Delta}w(p)u(c)\right)=\theta pc(1-\lambda)$

Inequality (19) can be rewritten as (21):

(21) $\quad \theta py(1-\lambda)<(1-\theta)\left(\delta^t w(\lambda p)u(y)-\delta^{t+\Delta}w(p)u(y)\right)$

From (20) and (21), we obtain

$$(1-\theta)\left(\delta^t w(\lambda p)u(c)-\delta^{t+\Delta}w(p)u(c)\right)y<(1-\theta)\left(\delta^t w(\lambda p)u(y)-\delta^{t+\Delta}w(p)u(y)\right)c.$$

For all $\theta\in(0,1)$, the above inequality reduces to, $u(c)/c<u(y)/y$, which, for a power utility function, $u(z)=z^\alpha$, with $z\geq 0$, becomes $\frac{c^\alpha}{c}<\frac{y^\alpha}{y}$ where we recall that $0<y<c$.

Taking the natural log of both sides confirms that the inequality holds for any $\alpha<1$.

(Necessity) If $\theta=0$ or $\theta=1$, the utilities cancel and subendurance does not hold. ■

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